MACMILLAN'S CANADIAN SCHOOL SERIES

A
SCHOOL GEOMETRY

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GEOMETRY

PART I

AXIOMS

ALL mathematical reasoning is founded on certain simple principles, the truth of which is so evident that they are accepted without proof. These self-evident truths are called **Axioms.**

For instance:

Things which are equal to the same thing are equal to one another.

The following axioms, corresponding to the first four Rules of Arithmetic, are among those most commonly used in geometrical reasoning.

Addition. *If equals are added to equals, the sums are equal.*

Subtraction. *If equals are taken from equals, the remainders are equal.*

Multiplication. *Things which are the same multiples of equals are equal to one another.*

For instance: *Doubles of equals are equal to one another.*

Division. *Things which are the same parts of equals are equal to one another.*

For instance: *Halves of equals are equal to one another.*

The above Axioms are given as instances, and not as a complete list, of those which will be used. They are said to

be *general*, because they apply equally to magnitudes of all kinds. Certain special axioms relating to geometrical magnitudes only will be stated from time to time as they are required.

Definitions and First Principles

Every beginner knows in a general way what is meant by a **point**, a **line**, and a **surface**. But in geometry these terms are used in a strict sense which needs some explanation.

1. A **point** has position, but is said to have *no magnitude*.

This means that we are to attach to a point no idea of size either as to *length* or *breadth*, but to think only where it is situated. A dot made with a sharp pencil may be taken as roughly representing a point; but small as such a dot may be, it still has *some* length and breadth, and is therefore not actually a geometrical point. The smaller the dot however, the more nearly it represents a point.

2. A **line** has length, but is said to have *no breadth*.

A line is traced out by a moving point. If the point of a pencil is moved over a sheet of paper, the trace left represents a line. But such a trace, however finely drawn, has some degree of breadth, and is therefore not itself a true geometrical line. The finer the trace left by the moving pencil-point, the more nearly will it represent a line.

3. Proceeding in a similar manner from the idea of a line to the idea of a surface, we say that

A **surface** has length and breadth, but *no thickness*.

And finally,

A **solid** has length, breadth, and thickness.

Solids, surfaces, lines, and points are thus related to one another:

(i) A solid is bounded by surfaces.

(ii) A surface is bounded by lines; and surfaces meet in lines.

(iii) A line is bounded (or terminated) by points; and lines meet in points.

4. A line may be straight or curved.

A **straight** line has the same direction from point to point throughout its whole length.

A **curved** line changes its direction continually from point to point.

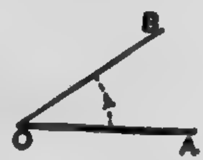
AXIOM. *There can be only one straight line joining two given points:* that is,

Two straight lines cannot enclose a space.

5. A **plane** is a *flat* surface, the test of flatness being that if any two points are taken in the surface, the straight line between them lies wholly in that surface.

6. When two straight lines meet at a point, they are said to form an **angle**.

The straight lines are called the **arms** of the angle; the point at which they meet is its **vertex**.



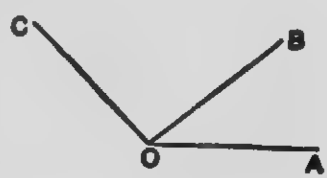
The magnitude of the angle may be thus explained:

Suppose that the arm OA is fixed, and that OB *turns about the point* O (as shewn by the arrow). Suppose also that OB began its turning from the position OA. Then the size of the angle AOB is measured by the *amount of turning* required to bring the revolving arm from its first position OA into its subsequent position OB.

Observe that the size of an angle does not in any way depend on the length of its arms.

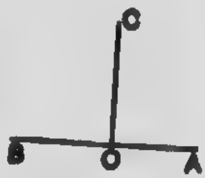
Angles which lie on either side of a common arm are said to be **adjacent**.

For example, the angles AOB, BOC, which have the common arm OB, are adjacent.



When two straight lines such as AB, CD cross one another at O, the angles COA, BOD are said to be **vertically opposite.** The angles AOD, COB are also vertically opposite to one another.

7. When one straight line stands on another so as to make the adjacent angles equal to one another, each of the angles is called a **right angle**; and each line is said to be **perpendicular** to the other.



Axioms. (i) *If O is a point in a straight line AB, then a line OC, which turns about O from the position OA to the position OB, must pass through one position, and only one, in which it is perpendicular to AB.*

(ii) *All right angles are equal.*

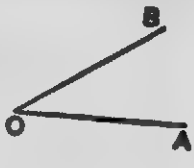
A right angle is divided into 90 equal parts called **degrees** (°); each degree into 60 equal parts called **minutes** (′); each minute into 60 equal parts called **seconds** (″).

In the above figure, if OC revolves about O from the position OA into the position OB, it turns through *two right angles*, or 180°.

If OC makes a *complete revolution* about O, starting from OA and returning to its original position, it turns through *four right angles*, or 360°.

8. An angle which is less than one right angle is said to be **acute.**

That is, an acute angle is less than 90°.



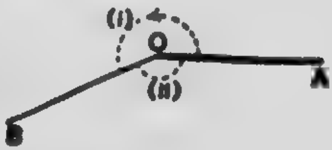
9. An angle which is greater than *one* right angle, but less than *two* right angles, is said to be **obtuse.**

That is, an obtuse angle lies between 90° and 180°.

10. If one arm OB of an angle turns until it makes *a straight line* with the other arm OA, the angle so formed is called a **straight angle.**

A straight angle = 2 right angles = 180°.

11. An angle which is greater than *two* right angles, but less than *four* right angles, is said to be **reflex.**

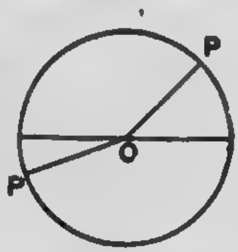


That is, a reflex angle lies between 180° and 360°.

NOTE. When two straight lines meet, *two* angles are formed, one greater, and one less than two right angles. The first arises by supposing OB to have revolved from the position OA the longer way round, marked (i); the other by supposing OB to have revolved the shorter way round, marked (ii). Unless the contrary is stated, the angle between two straight lines will be considered to be that which is less than two right angles.

12. Any portion of a plane surface bounded by one or more lines is called a **plane figure.**

13. A **circle** is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.



Here the point P moves so that its distance from the fixed point O is always the same.

The fixed point is called the **centre**, and the bounding line is called the **circumference.**

14. A **radius** of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal.

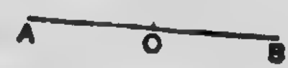
15. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

16. An **arc** of a circle is any part of the circumference.

17. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

18. To **bisect** means to divide into two *equal* parts.

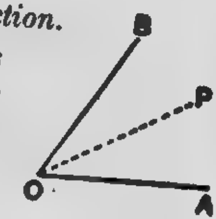
AXIOMS. (i) *If a point O moves from A to B along the straight line AB, it must pass through one position in which it divides AB into two equal parts.*



That is to say:

Every finite straight line has a point of bisection.

(ii) *If a line OP, revolving about O, turns from OA to OB, it must pass through one position in which it divides the angle AOB into two equal parts.*



That is to say:

Every angle may be supposed to have a line of bisection.

HYPOTHETICAL CONSTRUCTIONS

From the Axioms attached to Definitions 7 and 18, it follows that we may suppose

(i) *A straight line to be drawn perpendicular to a given straight line from any point in it.*

(ii) *A finite straight line to be bisected at a point.*

(iii) *An angle to be bisected by a line.*

SUPERPOSITION AND EQUALITY

AXIOM. *Magnitudes which can be made to coincide with one another are equal.*

This axiom implies that any line, angle, or figure may be taken up from its position, and without change in size or form, laid down

upon a second line, angle, or figure, for the purpose of comparison, and it states that two such magnitudes are equal when one can be exactly placed over the other without overlapping.

This process is called **superposition**, and the first magnitude is said to be **applied** to the other.

POSTULATES

In order to draw geometrical figures certain instruments are required. These are, for the purposes of this book, (i) *a straight ruler*, (ii) *a pair of compasses.* The following Postulates (or requests) claim the use of these instruments, and assume that with their help the processes mentioned below may be duly performed.

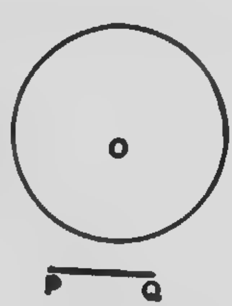
Let it be granted :

1. *That a straight line may be drawn from any one point to any other point.*

2. *That a* FINITE (*or terminated*) *straight line may be* PRODUCED (*that is, prolonged*) *to any length in that straight line.*

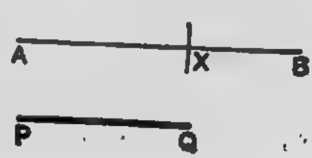
3. *That a circle may be drawn with any point as centre and with a radius of any length.*

NOTES. (i) Postulate 3, as stated above, implies that we may adjust the compasses to the length of any straight line PQ, and with a radius of this length draw a circle with any point O as centre. That is to say, the compasses may be used to *transfer distances* from one part of a diagram to another.



(ii) *Hence from AB, the greater of two straight lines, we may cut off a part equal to PQ the less.*

For if with centre A, and radius equal to PQ, we draw an arc of a circle cutting AB at X, it is obvious that AX is equal to PQ.



Introductory

1. Plane geometry deals with the properties of such lines and figures as may be drawn on a plane surface.

2. The subject is divided into a number of separate discussions, called **propositions.**

Propositions are of two kinds, **Theorems** and **Problems.**

A **Theorem** proposes to prove the truth of some geometrical statement.

A **Problem** proposes to perform some geometrical construction, such as to draw some particular line, or to construct some required figure.

3. A Proposition consists of the following parts:
The *General Enunciation*, the *Particular Enunciation*, the *Construction*, and the *Proof*.

(i) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition.

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily.

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem.

(iv) The **Proof** shews that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true.

4. The letters Q.E.D. are appended to a theorem, and stand for **Quod erat Demonstrandum,** *which was to be proved.*

5. A **Corollary** is a statement the truth of which follows readily from an established proposition; it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof.

6. The following symbols and abbreviations are used in the text of this book:

In Part I.

$\therefore$	*for* therefore,	$\angle$	*for* angle,
$=$	" is, or are, equal to,	$\triangle$	" triangle.

After Part I.

pt.	*for* point,	perp.	*for* perpendicular,
st. line	" straight line,	par^{m}	" parallelogram,
rt. $\angle$	" right angle,	rectil.	" rectilineal,
par^{l} (or $\parallel$)	" parallel,	$\odot$	" circle,
sq.	" square,	$\bigcirc^{ce}$	" circumference;

and all obvious contractions of commonly occurring words, such as opp., adj., diag., etc., *for* opposite, adjacent, diagonal, etc.

[For convenience of oral work, and to prevent the rather common abuse of contractions by beginners, the above code of signs has been introduced gradually, and at first somewhat sparingly.]

In numerical examples the following abbreviations will be used.

m.	*for* metre,	cm.	*for* centimetre,
mm.	" millimetre.	km.	" kilometre.

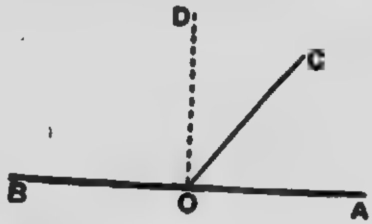
Also inches are denoted by the symbol (″).

Thus 5″ means 5 inches.

ON LINES AND ANGLES

THEOREM 1. [Euclid I. 13]

The adjacent angles which one straight line makes with another straight line on one side of it, are together equal to two right angles.



Let the straight line CO make with the straight line AB the adjacent $\angle s$ AOC, COB.

It is required to prove that the $\angle s$ AOC, COB are together equal to two right angles.

Suppose OD is at right angles to BA.

Proof. Then the $\angle s$ AOC, COB together

= the three $\angle s$ AOC, COD, DOB.

Also the $\angle s$ AOD, DOB together

= the three $\angle s$ AOC, COD, DOB.

$\therefore$ the $\angle s$ AOC, COB together = the $\angle s$ AOD, DOB

= two right angles.

Q.E.D.

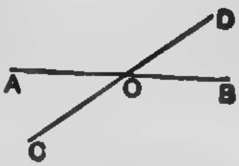
PROOF BY ROTATION

Suppose a straight line revolving about O turns from the position OA into the position OC, and thence into the position OB; that is, let the revolving line turn in succession through the $\angle s$ AOC, COB.

Now in passing from its first position OA to its final position OB, the revolving line turns through two right angles, for AOB is a straight line.

Hence the $\angle s$ AOC, COB together = two right angles.

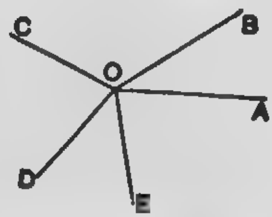
Corollary 1. *If two straight lines cut one another, the four angles so formed are together equal to four right angles.*



For example,

$\angle BOD + \angle DOA + \angle AOC + \angle COB = 4$ right angles.

Corollary 2. *When any number of straight lines meet at a point, the sum of the consecutive angles so formed is equal to four right angles.*



For a straight line revolving about O, and turning in succession through the $\angle s$ AOB, BOC, COD, DOE, EOA, will have made one complete revolution, and therefore turned through four right angles.

DEFINITIONS

(i) Two angles whose sum is *two* right angles are said to be **supplementary**; and each is called the **supplement** of the other.

Thus in the Fig. of Theor. 1 the angles AOC, COB are supplementary. Again the angle 123° is the supplement of the angle 57°.

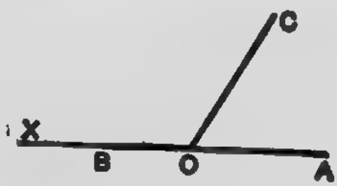
(ii) Two angles whose sum is *one* right angle are said to be **complementary**; and each is called the **complement** of the other.

Thus in the Fig. of Theor. 1 the angle DOC is the complement of the angle AOC. Again angles of 34° and 56° are complementary.

Corollary 3. (i) *Supplements of the same angle are equal.*
(ii) *Complements of the same angle are equal.*

THEOREM 2. [Euclid I. 14]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line.



At O in the straight line CO let the two straight lines OA, OB, on opposite sides of CO, make the adjacent $\angle s$ AOC, COB together equal to two right angles: (that is, let the adjacent $\angle s$ AOC, COB be supplementary).

It is required to prove that OB and OA are in the same straight line.

Produce AO beyond O to any point X: it will be shewn that OX and OB are the same line.

Proof. Since by construction AOX is a straight line,
$\therefore$ the $\angle$ COX is the supplement of the $\angle$ COA. *Theor.* 1
But, by hypothesis,
the $\angle$ COB is the supplement of the $\angle$ COA.
$\therefore$ the $\angle$ COX = the $\angle$ COB;
$\therefore$ OX and OB are the same line.

But, by construction, OX is in the same straight line with OA;

hence OB is also in the same straight line with OA.

Q.E.D.

EXERCISES

1. Write down the supplements of *one-half* of a right angle, *four-thirds* of a right angle; also of 46°, 149°, 83°, 101° 15′.

2. Write down the complement of *two-fifths* of a right angle; also of 27°, 38° 16′, and 41° 29′ 30″.

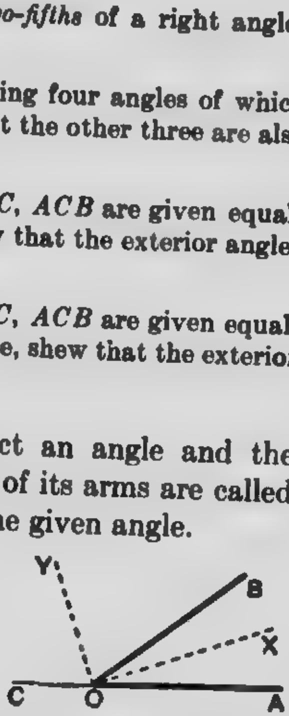
3. If two straight lines intersect forming four angles of which one is known to be a right angle, prove that the other three are also right angles.

4. In the triangle ABC the angles ABC, ACB are given equal. If the side BC is produced both ways, shew that the exterior angles so formed are equal.

5. In the triangle ABC the angles ABC, ACB are given equal. If AB and AC are produced beyond the base, shew that the exterior angles so formed are equal.

Definition. The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the **internal** and **external bisectors** of the given angle.

Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB.



6. Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle. That is to say, *the internal and external bisectors of an angle are at right angles to one another.*

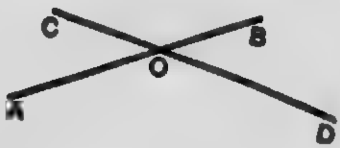
7. Shew that the angles AOX and COY in the above diagram are complementary.

8. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

9. If the angle AOB is 35°, find the angle COY.

THEOREM 3. [Euclid I. 15]

If two straight lines cut one another, the vertically opposite angles are equal.



Let the straight lines AB, CD cut one another at the point O.

It is required to prove that

(i) *the* $\angle AOC$ = *the* $\angle DOB$;

(ii) *the* $\angle COB$ = *the* $\angle AOD$.

Proof. Because AO meets the straight line CD,

∴ the adjacent $\angle$s AOC, AOD together = two right angles;

that is, the $\angle AOC$ is the supplement of the $\angle AOD$.

Again, because DO meets the straight line AB,

∴ the adjacent $\angle$s DOB, AOD together = two right angles;

that is, the $\angle DOB$ is the supplement of the $\angle AOD$.

Thus each of the $\angle$s AOC, DOB is the supplement of the $\angle AOD$,

∴ the $\angle AOC$ = the $\angle DOB$.

Similarly, the $\angle COB$ = the $\angle AOD$.

Q.E.D.

PROOF BY ROTATION

Suppose the line COD to revolve about O until OC turns into the position OA. Then at the same moment OD must reach the position OB (for AOB and COD are *straight*).

Thus the same amount of turning is required to close the $\angle AOC$ as to close the $\angle DOB$.

∴ the $\angle AOC$ = the $\angle DOB$.

EXERCISES ON ANGLES

(Numerical)

1. Through what angles does the minute-hand of a clock turn in (i) 5 minutes, (ii) 21 minutes, (iii) $43\frac{1}{3}$ minutes, (iv) 14 min. 10 sec.? And how long will it take to turn through (v) 66°, (vi) 222°?

2. A clock is started at noon: through what angles will the hour-hand have turned by (i) 3.45, (ii) 10 minutes past 5? And what will be the time when it has turned through $172\frac{1}{2}$°?

3. The earth makes a complete revolution about its axis in 24 hours. Through what angle will it turn in 3 hrs. 20 min.?

4. In the diagram of Theorem 3

(i) If the $\angle AOC = 35°$, write down (without measurement) the value of each of the $\angle$s COB, BOD, DOA.

(ii) If the $\angle$s COB, AOD together make up 250°, find each of the $\angle$s COA, BOD.

(iii) If the $\angle$s AOC, COB, BOD together make up 274°, find each of the four angles at O.

(Theoretical)

5. If from O, a point in AB, two straight lines OC, OD are drawn on opposite sides of AB so as to make the angle COB equal to the angle AOD; shew that OC and OD are in the same straight line.

6. Two straight lines AB, CD cross at O. If OX is the bisector of the angle BOD, prove that XO produced bisects the angle AOC.

7. Two straight lines AB, CD cross at O. If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.

8. If OX bisects an angle AOB, shew that, by folding the diagram about the bisector, OA may be made to coincide with OB.

How would OA fall with regard to OB, if

(i) the $\angle AOX$ were greater than the $\angle XOB$;

(ii) the $\angle AOX$ were less than the $\angle XOB$?

9. AB and CD are straight lines intersecting at right angles at O; shew by folding the figure about AB, that OC may be made to fall along OD.

10. A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB; shew that the crease left in the paper is perpendicular to AB.

ON TRIANGLES

1. Any portion of a plane surface bounded by one or more lines is called a **plane figure**.

The sum of the bounding lines is called the **perimeter** of the figure. The amount of surface enclosed by the perimeter is called the **area**.

2. **Rectilineal figures** are those which are bounded by straight lines.

3. A **triangle** is a plane figure bounded by *three* straight lines.

4. A **quadrilateral** is a plane figure bounded by *four* straight lines.

5. A **polygon** is a plane figure bounded by more than four straight lines.

6. A rectilineal figure is said to be

equilateral, when all its sides are equal;
equiangular, when all its angles are equal;
regular, when it is both equilateral and equiangular.

7. Triangles are thus classified with regard to their sides: A triangle is said to be

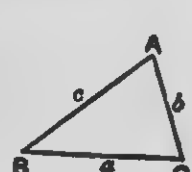
equilateral, when *all* its sides are equal;
isosceles, when *two* of its sides are equal;
scalene, when its sides are all unequal.

Equilateral Triangle

Isosceles Triangle

Scalene Triangle

In a triangle ABC, the letters A, B, C often denote the *magnitude* of the several angles (as measured in degrees); and the letters a, b, c the *lengths* of the opposite sides (as measured in inches, centimetres, or some other unit of length).



Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the **base**.

In an *isosceles* triangle the term *vertex* is usually applied to the point at which the equal sides intersect; and the vertical angle is the angle included by them.

8. Triangles are thus classified with regard to their angles: A triangle is said to be

right-angled, when one of its angles is a right angle;
obtuse-angled, when one of its angles is obtuse;
acute-angled, when *all three* of its angles are acute.

[It will be seen hereafter (Theorem 8. Cor. 1) *that every triangle must have at least two acute angles.*]

Right-angled Triangle — Obtuse-angled Triangle — Acute-angled Triangle

In a right-angled triangle the side opposite to the right angle is called the **hypotenuse**.

9. In any triangle the straight line joining a vertex to the middle point of the opposite side is called a **median**.

THE COMPARISON OF TWO TRIANGLES

(i) The three sides and three angles of a triangle are called its six **parts**. A triangle may also be considered with regard to its **area**.

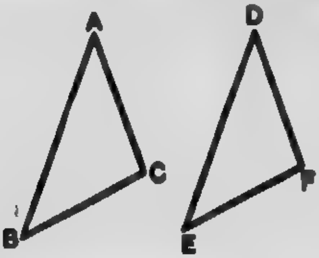
(ii) Two triangles are said to be **equal in all respects**, when one may be so placed upon the other as to exactly coincide with it; in which case each part of the first triangle is equal to the **corresponding** part (namely that with which it coincides) of the other; and the triangles are equal in area.

In two such triangles corresponding sides are *opposite to equal angles*, and corresponding angles are *opposite to equal sides.*

Triangles which may thus be made to coincide by superposition are said to be **identically equal or congruent.**

THEOREM 4. [Euclid I. 4]

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles included by those sides equal, then the triangles are equal in all respects.



Let ABC, DEF be two triangles in which

$$AB = DE,$$
$$AC = DF,$$

and the included angle BAC = the included angle EDF.

It is required to prove that the $\triangle ABC$ = *the* $\triangle DEF$ *in all respects.*

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$,
so that the point A falls on the point D,
and the side AB a[illegible]ng the side DE.
Then because $AB = DE$,
$\therefore$ the point B must coincide with the point E.
And because AB falls along DE,
and the $\angle BAC = \angle EDF$,
$\therefore$ AC must fall along DF.
And because $AC = DF$,
$\therefore$ the point C must coincide with the point F.
Then since B coincides with E, and C with F,
$\therefore$ the side BC must coincide with the side EF.
Hence the $\triangle ABC$ coincides with the $\triangle DEF$,
and is therefore equal to it in all respects.

Q.E.D.

Obs. In this Th. rem we must carefully observe what is *given* and what is *proved*.

$$\textit{Given that} \left\{ \begin{array}{r} AB = DE, \\ AC = DF, \\ \text{and the } \angle BAC = \text{the } \angle EDF. \end{array} \right.$$

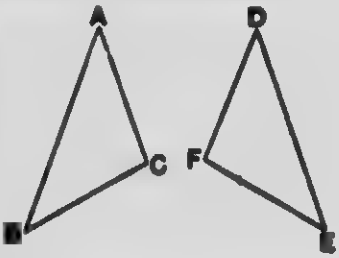
From these data we prove that the triangles coincide on superposition.

$$\text{Hence we } \textit{conclude} \text{ that} \left\{ \begin{array}{r} BC = EF, \\ \text{the } \angle ABC = \text{the } \angle DEF, \\ \text{and the } \angle ACB = \text{the } \angle DFE; \end{array} \right.$$

also that the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

NOTE. The adjoining diagram shows that in order to make two congruent triangles coincide, it may be necessary to *reverse*, that is, turn over one of them before superposition.



EXERCISES

1. *Shew that the bisector of the vertical angle of an isosceles triangle* (i) *bisects the base,* (ii) *is perpendicular to the base.*

2. *Let O be the middle point of a straight line AB, and let OC be perpendicular to it. If P is any point in OC, prove that $PA = PB$.*

3. Assuming that the four sides of a square $ABCD$ are equal, and that its angles are all right angles, prove the diagonals AC, BD equal.

4. $ABCD$ is a square, and L, M, and N are the middle points of AB, BC, and CD: using a separate figure in each case, prove that (i) $LM = MN$. (ii) $AM = DM$. (iii) $AN = AM$. (iv) $BN = DM$.

5. ABC is an isosceles triangle: from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that $BY = CX$.

THEOREM 5. [Euclid I. 5]

The angles at the base of an isosceles triangle are equal.

Let ABC be an isosceles triangle, in which the side $AB =$ the side AC.

It is required to prove that the $\angle ABC =$ *the* $\angle ACB$.

Suppose that AD is the line which bisects the $\angle BAC$, and let it meet BC in D.

1st Proof. Then in the $\triangle$s BAD, CAD,

because $\begin{cases} BA = CA, \\ AD \text{ is common to both triangles,} \\ \text{and the included } \angle BAD = \text{the included } \angle CAD; \end{cases}$

$\therefore$ the triangles are equal in all respects; *Theor.* 4.

so that the $\angle ABD =$ the $\angle ACD$.

Q.E.D.

2nd Proof. Suppose the $\triangle ABC$ to be folded about AD.

Then since the $\angle BAD =$ the $\angle CAD$,

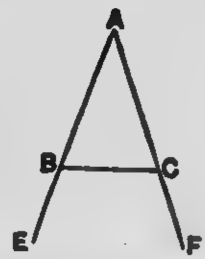
$\therefore AB$ must fall along AC.

And since $AB = AC$,

$\therefore B$ must fall on C, and consequently DB on DC.

$\therefore$ the $\angle ABD$ will coincide with the $\angle ACD$, and is therefore equal to it.

Q.E.D.



Corollary 1. *If the equal sides AB, AC of an isosceles triangle are produced, the exterior angles EBC, FCB are equal; for they are the supplements of the equal angles at the base.*

Corollary 2. *If a triangle is equilateral, it is also equiangular.*

Definition. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry**.

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

EXERCISES

1. $ABCD$ is a four-sided figure whose sides are all equal, and the diagonal BD is drawn: shew that

(i) the angle ABD = the angle ADB;
(ii) the angle CBD = the angle CDB;
(iii) the angle ABC = the angle ADC.

2. ABC, DBC are two isosceles triangles drawn on the same base BC, but on *opposite* sides of it: prove (by means of Theorem 5) that

the angle ABD = the angle ACD.

3. ABC, DBC are two isosceles triangles drawn on the same base BC and on *the same* side of it: employ Theorem 5 to prove that

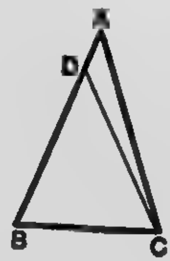
the angle ABD = the angle ACD.

4. AB, AC are the equal sides of an isosceles triangle ABC; and L, M, N are the middle points of AB, BC, and CA respectively: prove that (i) $LM = NM$. (ii) $BN = CL$.

(iii) the angle ALM = the angle ANM.

THEOREM 6. [Euclid I. 6]

If two angles of a triangle are equal to one another, then the sides which are opposite to the equal angles are equal to one another.



Let ABC be a triangle in which

the $\angle ABC$ = the $\angle ACB$.

It is required to prove that the side AC = the side AB.

If AC and AB are not equal, suppose that AB is the greater.

From BA cut off BD equal to AC.

Join DC.

Proof. Then in the $\triangle$s DBC, ACB,

because $\begin{cases} DB = AC, \\ BC \text{ is common to both,} \\ \text{and the included } \angle DBC = \text{the included } \angle ACB; \end{cases}$

$\therefore$ the $\triangle DBC$ = the $\triangle ACB$ in area, *Theor.* 4.

the part equal to the whole ; which is absurd.

$\therefore AB$ is not unequal to AC ;

that is, $AB = AC$.

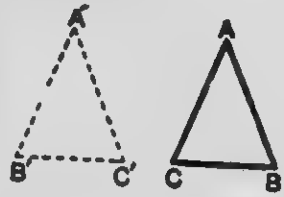
Q.E.D.

COROLLARY. *An equiangular triangle is also equilateral.*

In Theorem 6 we employ an *indirect method of proof* frequently used in geometry. It consists in shewing that the theorem *cannot be untrue;* since, if it were, we should be led to some *impossible conclusion.* This form of proof is known as **Reductio ad Absurdum.**

NOTE ON THEOREMS 5 AND 6

Theorems 5 and 6 may be verified experimentally by cutting out the given $\triangle ABC$, and, after turning it over, fitting it *thus reversed* into the vacant space left in the paper.



Suppose $A'B'C'$ to be the original position of the $\triangle ABC$, and let ACB represent the triangle when reversed.

In Theorem 5, it will be found on applying A to A' that C may be made to fall on B', and B on C'.

In Theorem 6, on applying C to B' and B to C' we find that A will fall on A'.

In either case the given triangle *reversed* will coincide with its own "trace," so that the side and angle on the *left* are respectively equal to the side and angle on the *right*.

NOTE ON A THEOREM AND ITS CONVERSE

The enunciation of a theorem consists of two clauses. The first clause tells us what we are to *assume*, and is called the **hypothesis**; the second tells us what *it is required to prove*, and is called the **conclusion**.

For example, the enunciation of Theorem 5 assumes that in a certain triangle ABC *the side* AB = *the side* AC: this is the *hypothesis*. From this it is required to prove that *the angle* ABC = *the angle* ACB: this is the *conclusion*.

If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first.

For example, in Theorem 5

it is *assumed* that $AB = AC$;
it is *required to prove* that the angle ABC = the angle ACB.

Now in Theorem 6

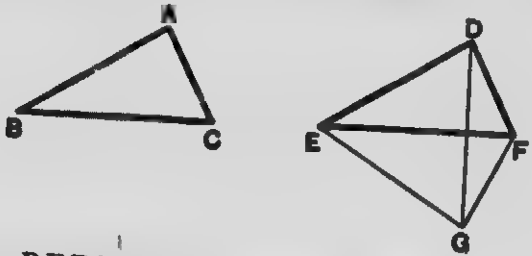
it is *assumed* that the angle ABC = the angle ACB;
it is *required to prove* that $AB = AC$.

Thus we see that Theorem 6 is the converse of Theorem 5; for the *hypothesis of each is the conclusion of the other*.

It must not however be supposed that if a theorem is true, its converse is *necessarily* true. [See p. 25.]

THEOREM 7. [Euclid I. 8]

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they are equal in all respects.



Let ABC, DEF be two triangles in which

$$AB = DE,$$
$$AC = DF,$$
$$BC = EF.$$

It is required to prove that the triangles are equal in all respects.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$, so that B falls on E, and BC along EF, and so that A is on the side of EF opposite to D.

Then because $BC = EF$, C must fall on F.

Let GEF be the new position of the $\triangle ABC$.

Join DG.

Because $ED = EG$,

$\therefore$ the $\angle EDG =$ the $\angle EGD$. *Theor.* 5.

Again, because $FD = FG$,

$\therefore$ the $\angle FDG =$ the $\angle FGD$.

Hence the whole $\angle EDF =$ the whole $\angle EGF$;

that is, the $\angle EDF =$ the $\angle BAC$.

Then in the $\triangle$s BAC, EDF;

because $\begin{cases} BA = ED, \text{ and } AC = DF, \\ \text{and the included } \angle BAC = \text{the included } \angle EDF; \end{cases}$

$\therefore$ the triangles are equal in all respects. *Theor.* 4.

Q.E.D.

Obs. In this Theorem

it is *given* that $AB = DE$, $BC = EF$, $CA = FD$;
and we *prove* that $\angle C = \angle F$, $\angle A = \angle D$, $\angle C = \angle E$.
Also the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

NOTE 1. We have taken the case in which DG falls within the $\triangle$s EDF, EGF.

Two other cases might arise:
(i) DG might fall outside the $\triangle$s EDF, EGF [as in Fig. 1.]
(ii) DG might coincide with DF, FG [as in Fig. 2.]

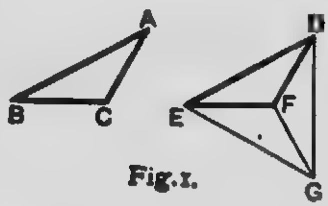


Fig. 1.

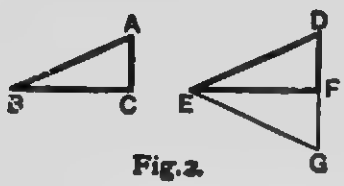


Fig. 2.

These cases will arise only when the given triangles are obtuse-angled or right-angled; and (as will be seen hereafter) not even then, if we begin by choosing for superposition the *greatest* side of the $\triangle ABC$, as in the diagram of page 24.

NOTE 2. Two triangles are said to be **equiangular to one another** when the angles of one are respectively equal to the angles of the other.

Hence if two triangles have the three sides of one severally equal to the three sides of the other, the triangles are equiangular to one another.

The student should state the converse theorem, and shew by a diagram that the converse is not necessarily true.

*** *At this stage Problems* 1-5 *and* 8 [*see page* 70] *may conveniently be taken, the proofs affording good illustrations of the Identical Equality of Two Triangles.*

EXERCISES

On the Identical Equality of Two Triangles
Theorems 4 and 7

(Theoretical)

1. Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base,
(i) bisects the vertical angle: (ii) is perpendicular to the base.

2. If $ABCD$ is a rhombus, that is, an equilateral foursided figure; shew, by drawing the diagonals AC, BD, that
(i) the angle ABC = the angle ADC;
(ii) AC bisects each of the angles BAD, BCD.
(iii) the diagonals bisect one another at right angles.

3. If in a quadrilateral $ABCD$ the opposite sides are equal, namely $AB = CD$ and $AD = CB$; prove that the angle ADC = the angle ABC.

4. If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of Theorem 7) that the angle ABD = the angle ACD, taking (i) the case where the triangles are on the *same* side of BC, (ii) the case where they are on *opposite* sides of BC.

5. If ABC, DBC are two isosceles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be divided into two equal parts.

6. Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal to one another.

7. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base; shew that they are also equidistant from the vertex.

8. Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

9. ABC is an isosceles triangle having AB equal to AC; and the angles at B and C are bisected by BO and CO: shew that
(i) $BO = CO$; (ii) AO bisects the angle BAC.

10. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF; and FB, EC are joined: shew that FB is equal to EC.

EXERCISES ON TRIANGLES

(Numerical and Graphical)

1. Draw a triangle ABC, having given $a = 2{\cdot}0''$, $b = 2{\cdot}1''$, $c = 1{\cdot}3''$. Measure the angles, and find their sum.

2. In the triangle ABC, $a = 7{\cdot}5$ cm., $b = 7{\cdot}0$ cm., $c = 6{\cdot}5$ cm. Draw and measure the perpendicular from B on CA.

3. Draw a triangle ABC, in which $a = 7$ cm., $b = 6$ cm., $C = 65°$.

How would you prove theoretically that any two triangles having these parts are alike in size and shape? Invent some experimental illustration.

4. Draw a triangle from the following data: $b = 2''$, $c = 2{\cdot}5''$, $A = 57°$; and measure a, B, and C.

Draw a second triangle, using as data the values just found for a, B, C; and measure b, c, A. What conclusion do you draw?

5. When the sun is 42° above the horizon, a vertical pole casts a shadow 30 ft. long. Represent this on a diagram (scale 1″ to 10 ft.); and find by measurement the approximate height of the pole.

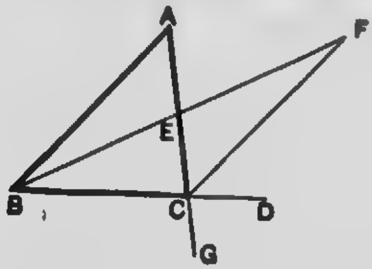
6. From a point A a surveyor goes 150 yards due East to B; then 300 yards due North to C; finally 450 yards due West to D. Plot his course (scale 1″ to 100 yards); and find roughly how far D is from A. Measure the angle DAB, and say in what direction D bears from A.

7. B and C are two points, known to be 260 yards apart, on a straight shore. A is a vessel at anchor. The angles CBA, BCA are observed to be 33° and 81° respectively. Find graphically the approximate distance of the vessel from the points B and C, and from the nearest point on shore.

8. In surveying a park it is required to find the distance between two points A and B; but as a lake intervenes, a direct measurement cannot be made. The surveyor therefore takes a third point C, from which both A and B are accessible, and he finds $CA = 245$ yards, $CB = 320$ yards, and the angle $ACB = 42°$. Ascertain from a plan the approximate distance between A and B.

THEOREM 8. [Euclid I. 16]

If one side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D.

It is required to prove that the exterior $\angle ACD$ is greater than either of the interior opposite $\angle$s ABC, BAC.

Suppose E to be the middle point of AC.

Join BE; and produce it to F, making EF equal to BE.

Join FC.

Proof. Then in the $\triangle$s AEB, CEF,

because $\left\{\begin{array}{l} AE = CE, \\ EB = EF, \\ \text{and the } \angle AEB = \text{the vertically opposite } \angle CEF; \end{array}\right.$

$\therefore$ the triangles are equal in all respects; *Theor.* 4.

so that the $\angle BAE$ = the $\angle ECF$.

But the $\angle ECD$ is greater than the $\angle ECF$;

$\therefore$ the $\angle ECD$ is greater than the $\angle BAE$;

that is, the $\angle ACD$ is greater than the $\angle BAC$.

In the same way, if AC is produced to G, by supposing A to be joined to the middle point of BC, it may be proved that the $\angle BCG$ is greater than the $\angle ABC$.

But the $\angle BCG$ = the vertically opposite $\angle ACD$.

$\therefore$ the $\angle ACD$ is greater than the $\angle ABC$. Q.E.D.

Corollary 1. *Any two angles of a triangle are together less than two right angles.*

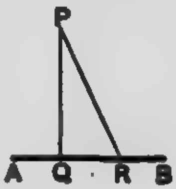
For the $\angle ABC$ is less than the $\angle ACD$: *Proved.*
to each add the $\angle ACB$.
Then the $\angle s$ ABC, ACB are less than the $\angle s$ ACD, ACB,
therefore, less than two right angles.

Corollary 2. *Every triangle must have at least two acute angles.*

For if one angle is obtuse or a right angle, then by Cor. 1 each of the other angles must be less than a right angle.

Corollary 3. *Only one perpendicular can be drawn to a straight line from a given point outside it.*

If two perpendiculars could be drawn to AB from P, we should have a triangle PQR in which each of the $\angle s$ PQR, PRQ would be a right angle, which is impossible.



EXERCISES

1. Prove Corollary 1 by joining the vertex A to any point in the base BC.

2. *ABC is a triangle and D any point within it. If BD and CD are joined, the angle BDC is greater than the angle BAC.* Prove this
   (i) by producing BD to meet AC.
   (ii) by joining AD, and producing it towards the base.

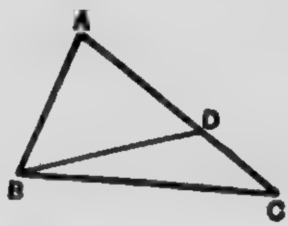
3. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

4. To a given straight line there cannot be drawn from a point outside it more than two straight lines of the same given length.

5. If the equal sides of an isosceles triangle are produced, the exterior angles must be obtuse.

THEOREM 9. [Euclid I. 18]

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less.



Let ABC be a triangle, in which the side AC is greater than the side AB.

It is required to prove that the $\angle ABC$ *is greater than the* $\angle ACB$.

From AC cut off AD equal to AB.
Join BD.

Proof. Because $AB = AD$,
$\therefore$ the $\angle ABD$ = the $\angle ADB$. *Theor.* 5.

But the exterior $\angle ADB$ of the $\triangle BDC$ is greater than the interior opposite $\angle DCB$; that is, greater than the $\angle ACB$.

$\therefore$ the $\angle ABD$ is greater than the $\angle ACB$.

Still more then is the $\angle ABC$ greater than the $\angle ACB$.

Q.E.D.

Obs. The mode of demonstration used in the following Theorem is known as the **Proof by Exhaustion.** It is applicable to cases in which *one* of certain suppositions must necessarily be true; and it consists in shewing that each of these suppositions is false *with one exception:* hence the truth of the remaining supposition is inferred.

THEOREM 10. [Euclid I. 19]

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less.

Let ABC be a triangle, in which the $\angle\ ABC$ is greater than the $\angle\ ACB$.

It is required to prove that the side AC is greater than the side AB.

Proof. If AC is not greater than AB,
it must be either equal to, or less than AB.

Now if AC were equal to AB,
then the $\angle\ ABC$ would be equal to the $\angle\ ACB$; *Theor.* 5.
but, by hypothesis, it is not.

Again, if AC were less than AB,
then the $\angle\ ABC$ would be less than the $\angle\ ACB$; *Theor.* 9.
but, by hypothesis, it is not.

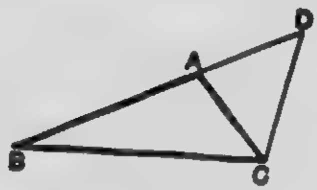
That is, AC is neither equal to, nor less than AB.

$\therefore$ AC is greater than AB. Q.E.D.

[For Exercises on Theorems 9 and 10 see page 34.]

Theorem 11. [Euclid I. 20]

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

It is required to prove that any two of its sides are together greater than the third side.

It is enough to shew that if BC is the greatest side, then BA, AC are together greater than BC.

Produce BA to D, making AD equal to AC.

Join DC.

Proof. Because $AD = AC$,

$\therefore$ the $\angle ACD$ = the $\angle ADC$. *Theor.* 5.

But the $\angle BCD$ is greater than the $\angle ACD$;

$\therefore$ the $\angle BCD$ is greater than the $\angle ADC$,

that is, than the $\angle BDC$.

Hence from the $\triangle BDC$,

BD is greater than BC. *Theor.* 10.

But $BD = BA$ and AC together;

$\therefore$ BA and AC are together greater than BC.

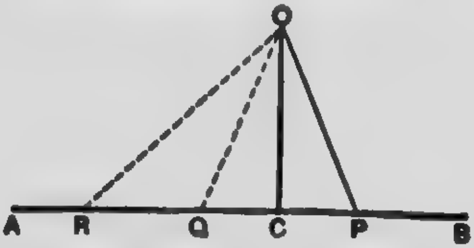
Q.E.D.

Note. This proof may serve as an exercise. But the truth of the Theorem is really self-evident. For to go from B to C along the straight line BC is clearly shorter than to go from B to A and then from A to C. In other words

The shortest distance between two points is the straight line which joins them.

THEOREM 12

Of all straight lines drawn from a given point to a given straight line the perpendicular is the least.



Let OC be the perpendicular, and OP *any* oblique, drawn from the given point O to the given straight line AB.

It is required to prove that OC is less than OP.

Proof. In the $\triangle\, OCP$, since the $\angle\, OCP$ is a right angle,

$\therefore$ the $\angle\, OPC$ is less than a right angle; *Theor.* 8. *Cor.*

that is, the $\angle\, OPC$ is less than the $\angle\, OCP$.

$\therefore$ OC is less than OP. *Theor.* 10.

Q.E.D.

COROLLARY 1. Hence conversely, since there can be only one perpendicular and one shortest line from O to AB,

If OC is the shortest straight line from O to AB, then OC is perpendicular to AB.

COROLLARY 2. *Two obliques OP, OQ, which cut AB at equal distances from C, the foot of the perpendicular, are equal.*

The ▲ OCP, OCQ may be shewn to be congruent by Theorem 4; hence $OP = OQ$.

COROLLARY 3. *Of two obliques OQ, OR, if OR cuts AB at the greater distance from C, the foot of the perpendicular, then OR is greater than OQ.*

The $\angle\, OQC$ is acute, $\therefore$ the $\angle\, OQR$ is obtuse;

$\therefore$ the $\angle\, OQR$ is greater than the $\angle\, ORQ$;

$\therefore$ OR is greater than OQ.

EXERCISES ON INEQUALITIES IN A TRIANGLE

1. *The hypotenuse is the greatest side of a right-angled triangle.*

2. *The greatest side of any triangle makes acute angles with each of the other sides.*

3. *If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle.*

4. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

5. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

6. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

7. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

8. The difference of any two sides of a triangle is less than the third side.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. ABC is a triangle, and the vertical angle BAC is bisected by a line which meets BC in X; shew that BA is greater than BX, and CA greater than CX. Hence obtain a proof of Theorem 11.

11. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

12. The sum of the diagonals of a quadrilateral is not greater than the sum of the four straight lines drawn from the angular points to any given point. In what case are these sums equal?

13. *In a triangle any two sides are together greater than twice the median which bisects the remaining side.*

[Produce the median, and complete the construction after the manner of Theorem 8.]

14. *In any triangle the sum of the medians is less than the perimeter.*

PARALLELS

DEFINITION. **Parallel straight lines** are such as, being in the same plane, do not meet however far they are produced beyond both ends.

NOTE. Parallel lines must be *in the same plane*. For instance, two straight lines, one of which is drawn on a table and the other on the floor, would never meet if produced; but they are not for that reason necessarily parallel.

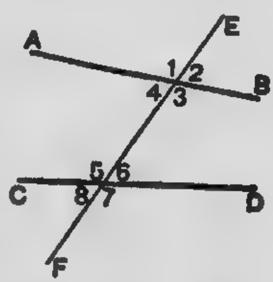
AXIOM. *Two intersecting straight lines cannot both be parallel to a third straight line.*

In other words:

Through a given point there can be only one straight line parallel to a given straight line.

This assumption is known as *Playfair's Axiom.*

DEFINITION. When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.

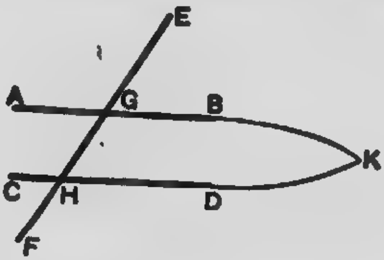


Thus in the adjoining figure,
1, 2 7, 8 are called **exterior angles**,
3, 4, 5, 6 are called **interior angles**,
4 and 6 are said to be **alternate angles**; so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the **exterior angle**, and 6 as the **interior opposite angle on the same side** of EF. Such angles are also known as **corresponding angles.** Similarly 7 and 3, 8 and 4, 1 and 5 are pairs of corresponding angles.

THEOREM 13. [Euclid I. 27 and 28]

If a straight line cuts two other straight lines so as to make
(i) *the alternate angles equal,*
or (ii) *an exterior angle equal to the interior opposite angle on the same side of the cutting line,*
or (iii) *the interior angles on the same side equal to two right angles;*
then in each case the two straight lines are parallel.

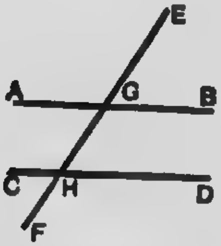


(i) Let the straight line $EGHF$ cut the two straight lines AB, CD at G and H so as to make the alternate $\angle s$ AGH, GHD equal to one another.

It is required to prove that AB and CD are parallel.

Proof. If AB and CD are not parallel, they will meet, if produced, either towards B and D, or towards A and C.
If possible, let AB and CD, when produced, meet towards B and D, at the point K.
Then KGH is a triangle, of which one side KG is produced to A;
$\therefore$ the exterior $\angle AGH$ is greater than the interior opposite $\angle GHK$; but, by hypothesis, it is not greater.
$\therefore$ AB and CD cannot meet when produced towards B and D.
Similarly it may be shewn that they cannot meet towards A and C:

$\therefore$ AB and CD are parallel.



(ii) Let the exterior $\angle EGB$ = the interior opposite $\angle GHD$.

It is required to prove that AB and CD are parallel.

Proof. Because the $\angle EGB$ = the $\angle GHD$,
and the $\angle EGB$ = the vertically opposite $\angle AGH$;
$\therefore$ the $\angle AGH$ = the $\angle GHD$:
and these are alternate angles;
$\therefore$ AB and CD are parallel.

(iii) Let the two interior $\measuredangle$ BGH, GHD be together equal to two right angles.

It is required to prove that AB and CD are parallel.

Proof. Because the $\measuredangle$ BGH, GHD together = two right angles;
and because the adjacent $\measuredangle$ BGH, AGH together = two right angles;
$\therefore$ the $\measuredangle$ BGH, AGH together = $\measuredangle$ BGH, GHD.
From these equals take the $\angle BGH$;
then the remaining $\angle AGH$ = the remaining $\angle GHD$:
and these are alternate angles;
$\therefore$ AB and CD are parallel. Q.E.D.

DEFINITION. A straight line drawn across a set of given lines is called a **transversal**.

For instance, in the above diagram the line $EGHF$, which crosses the given lines AB, CD, is a transversal.

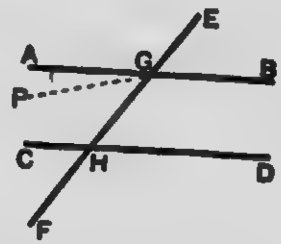
THEOREM 14. [Euclid I. 29]

If a straight line cuts two parallel lines, it makes

(i) *the alternate angles equal to one another;*

(ii) *the exterior angle equal to the interior opposite angle on the same side of the cutting line;*

(iii) *the two interior angles on the same side together equal to two right angles.*



Let the straight lines AB, CD be parallel, and let the straight line $EGHF$ cut them.

It is required to prove that

(i) *the* $\angle\ AGH$ = *the alternate* $\angle\ GHD$;

(ii) *the exterior* $\angle\ EGB$ = *the interior opposite* $\angle\ GHD$;

(iii) *the two interior* $\angle$s BGH, GHD *together = two right angles.*

Proof. (i) If the $\angle\ AGH$ is not equal to the $\angle\ GHD$, suppose the $\angle\ PGH$ equal to the $\angle\ GHD$, and alternate to it;

then PG and CD are parallel. *Theor.* 13.

But, by hypothesis, AB and CD are parallel;

$\therefore$ the two intersecting straight lines AG, PG are both parallel to CD: which is impossible. *Playfair's Axiom.*

$\therefore$ the $\angle\ AGH$ is not unequal to the $\angle\ GHD$;

that is, the alternate $\angle$s AGH, GHD are equal.

(ii) Again, because the $\angle EGB$ = the vertically opposite $\angle AGH$;

and the $\angle AGH$ = the alternate $\angle GHD$; *Proved.*

$\therefore$ the exterior $\angle EGB$ = the interior opposite $\angle GHD$.

(iii) Lastly, the $\angle EGB$ = the $\angle GHD$; *Proved.*

add to each the $\angle BGH$;

then the $\angle$s EGB, BGH together = the angles BGH, GHD.

But the adjacent $\angle$s EGB, BGH together = two right angles;

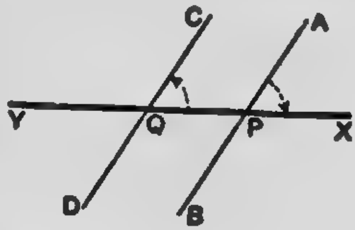
$\therefore$ the two interior $\angle$s BGH, GHD together = two right angles.

Q.E.D.

PARALLELS ILLUSTRATED BY ROTATION

The direction of a straight line is determined by the angle which it makes with some given line of reference.

Thus the direction of AB, relatively to the given line YX, is given by the angle APX.



Now suppose that AB and CD in the adjoining diagram are parallel; then we have learned that the ext. $\angle APX$ = the int. opp. $\angle CQX$; that is, AB and CD *make equal angles* with the line of reference YX.

This brings us to the leading idea connected with parallels:

Parallel straight lines have the same DIRECTION, *but differ in* POSITION.

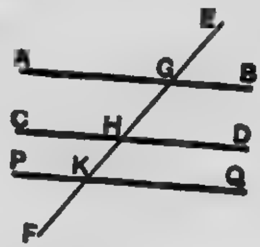
The same idea may be illustrated thus:

Suppose AB to rotate about P through the $\angle APX$, so as to take the position XY. Thence let it rotate about Q *the opposite way* through the equal $\angle XQC$: it will now take the position CD. Thus AB may be brought into the position of CD by two rotations which, being equal and opposite, involve no final change of direction.

Obs. If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite **senses** of the line AB.

THEOREM 15. [Euclid I. 30]

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ.

It is required to prove that AB and CD are parallel to one another.

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,

$\therefore$ the $\angle AGK$ = the alternate $\angle GKQ$.

And because CD and PQ are parallel, and EF meets them,

$\therefore$ the exterior $\angle GHD$ = the interior opposite $\angle GKQ$.

$\therefore$ the $\angle AGH$ = the $\angle GHD$;

and these are alternate angles;

$\therefore$ AB and CD are parallel. Q.E.D.

HYPOTHETICAL CONSTRUCTION. In the diagram on p. 39 let AB be a fixed straight line, Q a fixed point, CD a straight line turning about Q, and $YQPX$ any transversal through Q. Then as CD rotates, there must be one position in which the $\angle CQX$ = the fixed $\angle APX$.

Hence through any given point we may assume a line to pass parallel to any given straight line.

EXERCISES ON PARALLELS

1. In the diagram of the previous page, if the angle EGB is 55°, express in degrees each of the angles GHC, HKQ, QKF.

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.*

3. *If a straight line meets two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.*

4. *Angles of which the arms are parallel, each to each, are either equal or supplementary.*

5. Two straight lines AB, CD bisect one another at O. Shew that the straight lines joining AC and BD are parallel.

6. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

7. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

8. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

9. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

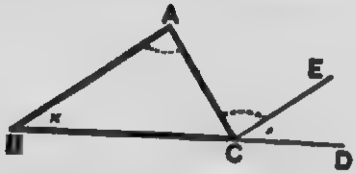
10. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

11. AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected: if through any point X in DC a straight line YXZ is drawn parallel to AB and meeting the bisectors in Y and Z, shew that XY is equal to XZ.

12. Two straight rods PA, QB revolve about pivots at P and Q, PA making 12 complete revolutions a minute, and QB making 10. If they start parallel and pointing the same way, how long will it be before they are again parallel, (i) pointing opposite ways, (ii) pointing the same way?

THEOREM 16. [Euclid I. 32]

The three angles of a triangle are together equal to two right angles.



Let ABC be a triangle.

It is required to prove that the three $\angle$s *ABC, BCA, CAB together = two right angles.*

Produce BC to any point D; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,

$\therefore$ the $\angle\, ACE$ = the alternate $\angle\, CAB$.

Again, because BA and CE are parallel, and BD meets them,

$\therefore$ the exterior $\angle\, ECD$ = the interior opposite $\angle\, ABC$.

$\therefore$ *the whole exterior* $\angle\, ACD$ = *the sum of the two interior opposite* $\angle$s *CAB, ABC.*

To each of these equals add the $\angle\, BCA$; then the $\angle$s BCA, ACD together = the three $\angle$s BCA, CAB, ABC.

But the adjacent $\angle$s BCA, ACD together = two right angles.

$\therefore$ the $\angle$s BCA, CAB, ABC together = two right angles.

Q.E.D.

Obs. In the course of this proof the following most important property has been established.

If a side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.

Namely, the ext. $\angle\, ACD$ = the $\angle\, CAB$ + the $\angle\, ABC$.

INFERENCES FROM THEOREM 16

1. *If A, B, and C denote the number of degrees in the angles of a triangle,*

$$\textit{then } A + B + C = 180°.$$

2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.*

3. *In any right-angled triangle the two acute angles are complementary.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

EXERCISES ON THEOREM 16

1. Each angle of an equilateral triangle is 60°.

2. In a right-angled isosceles triangle the angles are 45°, 45°, 90°.

3. Two angles of a triangle are 36° and 123° respectively: deduce the third angle; and verify your result by measurement.

4. In a triangle ABC, the $\angle B = 111°$, the $\angle C = 42°$; deduce the $\angle A$, and verify by measurement.

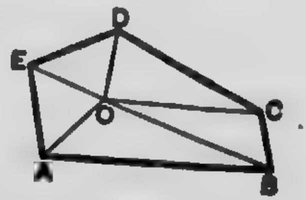
5. One side BC of a triangle ABC is produced to D. If the exterior angle ACD is 134°, and the angle BAC is 42°, find each of the remaining interior angles.

6. In the figure of Theorem 16, if the $\angle ACD = 118°$, and the $\angle B = 51°$, find the $\angle$s A and C; and check your results by measurement.

7. *Prove that $A + B + C = 180°$ by supposing a line drawn through the vertex parallel to the base.*

8. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

COROLLARY 1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Let $ABCDE$ be a rectilineal figure of n sides.

It is required to prove that

all the interior angles + 4 rt. ∠s = $2n$ rt. ∠s.

Take any point O within the figure, and join O to each of its vertices.

Then the figure is divided into n triangles.

And the three ∠s of each △ together = 2 rt. ∠s.

Hence *all* the ∠s of *all* the △s together = $2n$ rt. ∠s.

But all the ∠s of all the △s make up all the interior angles of the figure together with the angles at O, which = 4 rt. ∠s.

∴ all the int. ∠s of the figure + 4 rt. ∠s = $2n$ rt. ∠s.

Q.E.D.

DEFINITION. A **regular** polygon is one which has all its sides equal and all its angles equal.

Thus if D denotes the number of degrees in each angle of a *regular* polygon of n sides, the above result may be stated thus:

$$nD + 360° = n \cdot 180°.$$

EXAMPLE

Find the number of degrees in each angle of a regular

(i) hexagon (6 sides); (ii) octagon (8 sides); (iii) decagon (10 sides).

EXERCISES ON THEOREM 16

(Numerical and Graphical)

1. ABC is a triangle in which the angles at B and C are respectively double and treble of the angle at A: find the number of degrees in each of these angles.

2. The base of a triangle is produced both ways, and the exterior angles are found to be 94° and 126°; deduce the vertical angle. Construct such a triangle, and check your result by measurement.

3. The sum of the angles at the base of a triangle is 162°, and their difference is 60°: find all the angles.

4. The angles at the base of a triangle are 84° and 62°; deduce (i) the vertical angle, (ii) the angle between the bisectors of the base angles. Check your results by construction and measurement.

5. In a triangle ABC, the angles at B and C are 74° and 62°; if AB and AC are produced, deduce the angle between the bisectors of the exterior angles. Check your result graphically.

6. Three angles of a quadrilateral are respectively 114½°, 50°, and 75½°; find the fourth angle.

7. In a quadrilateral $ABCD$, the angles at B, C, and D are respectively equal to $2A$, $3A$, and $4A$; find the angles.

8. Four angles of an irregular pentagon (5 sides) are 40°, 78°, 122°, and 135°; find the fifth angle.

9. In any *regular* polygon of n sides, each angle contain $\frac{2(n-2)}{n}$ right angles.

   (i) Deduce this result from the Enunciation of Corollary 1.

   (ii) Prove it independently by joining one vertex A to each of the others (except the two immediately adjacent to A), thus dividing the polygon into n–2 triangles.

10. How many sides have the regular polygons each of whose angles is (i) 108°, (ii) 156°?

11. Shew that the only *regular* figures which may be fitted together so as to form a plane surface are (i) *equilateral triangles*, (ii) *squares*, (iii) *regular hexagons*.

Corollary 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*

1st Proof. Suppose, as before, that the figure has n sides. Now at *each* vertex

the interior $\angle$ + the exterior $\angle$ = 2 rt. $\angle$s.

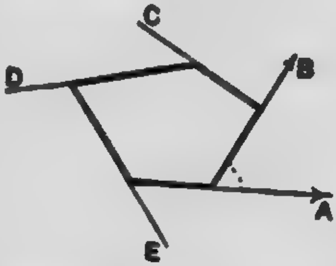
$\therefore$ the sum of the n int. $\angle$s + the sum of the n ext. $\angle$s = $2n$ rt. $\angle$s.

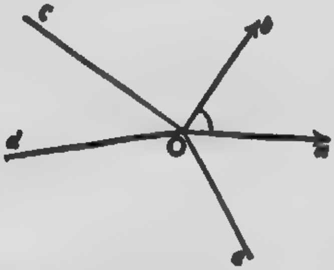
But by Corollary 1,

the sum of the int. $\angle$s + 4 rt. $\angle$s = $2n$ rt. $\angle$s;

$\therefore$ the sum of the ext. $\angle$s = 4 rt. $\angle$s.

2nd Proof. Q.E.D.





Take any point O, and suppose Oa, Ob, Oc, Od, and Oe are lines parallel to the sides marked A, B, C, D, E (and drawn from O in the *sense* in which those sides were produced).

Then the ext. $\angle$ between the sides A and B = the $\angle\, aOb$.

The other ext. $\angle$s = the respective $\angle$s bOc, cOd, dOe, eOa.

$\therefore$ the sum of the ext. $\angle$s = the sum of the $\angle$s at O

= 4 rt. $\angle$s.

EXERCISES

1. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the interior angle of an equilateral triangle.

2. Express in degrees the magnitude of each *exterior* angle of (i) a regular octagon, (ii) a regular decagon.

3. How many sides has a regular polygon if each exterior angle is (i) 30°, (ii) 24°?

4. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, shew that the bisectors meet at right angles.

5. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles *minus* the vertical angle is equal to two right angles.

6. In a triangle ABC the base angles at B and C are bisected by BO and CO respectively. Shew that the angle $BOC = 90° + \frac{A}{2}$.

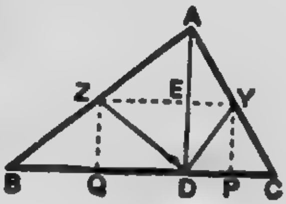
7. In the triangle ABC, the sides AB, AC are produced, and the exterior angles are bisected by BO and CO. Shew that the angle $BOC = 90° - \frac{A}{2}$.

8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

9. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

EXPERIMENTAL PROOF OF THEOREM 16

$[A + B + C = 180°]$



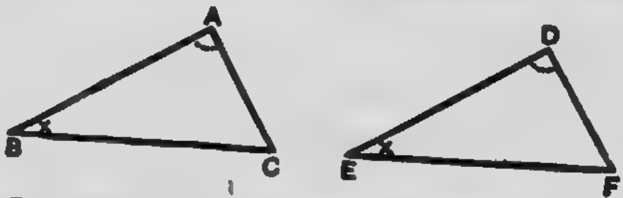
In the $\triangle ABC$, AD is perp. to BC, the greatest side. AD is bisected at right angles by ZY; and YP, ZQ are perps. on BC.

If now the $\triangle$ is folded about the three dotted lines, the $\angle$s A, B, and C will coincide with the $\angle$s ZDY, ZDQ, YDP;

$\therefore$ their sum is 180°.

THEOREM 17. [Euclid I. 26]

If two triangles have two angles of one equal to two angles of the other, each to each, and any side of the first equal to the corresponding side of the other, the triangles are equal in all respects.



Let ABC, DEF be two triangles in which

the $\angle A =$ the $\angle D$,

the $\angle B =$ the $\angle E$,

and the side $BC =$ the corresponding side EF.

It is required to prove that the $\triangle$s ABC, DEF *are equal in all respects.*

Proof. The sum of the $\angle$s $A, B, C = 2$ rt. $\angle$s *Theor.* 16.

$=$ the sum of the $\angle$s D, E, and F;

and the $\angle$s A and $B =$ the $\angle$s D and E respectively,

$\therefore$ the $\angle C =$ the $\angle F$.

Apply the $\triangle ABC$ to the $\triangle DEF$, so that B falls on E, and BC along EF.

Then, because $BC = EF$, C must coincide with F.

Because the $\angle B =$ the $\angle E$, BA must fall along ED.

And because the $\angle C =$ the $\angle F$, CA must fall along FD.

$\therefore$ the point A, which falls both on ED and on FD, must coincide with D, the point in which these lines intersect.

$\therefore$ the $\triangle ABC$ coincides with the $\triangle DEF$, and is therefore equal to it in all respects.

So that $AB = DE$, and $AC = DF$;

and the $\triangle DBC =$ the $\triangle DEF$ in area. Q.E.D.

EXERCISES

On the Identical Equality of Triangles

1. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

2. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

3. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B: shew that AX is equal to BY.

4. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

5. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

6. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of Theorem 8.]

7. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

8. A straight line drawn between two parallels, and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels is also bisected at that point.

9. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

10. *A surveyor wishes to ascertain the breadth of a river which he cannot cross. Standing at a point A near the bank, he notes an object B immediately opposite on the other bank. He lays down a line AC of any length at right angles to AB, fixing a mark at O, the middle point of AC. From C he walks along a line perpendicular to AC until he reaches a point D from which O and B are seen in the same direction. He now measures CD: prove that the result gives him the width of the river.*

ON THE IDENTICAL EQUALITY OF TRIANGLES

Three cases of the congruence of triangles have been dealt with in Theorems 4, 7, 17, the results of which are:

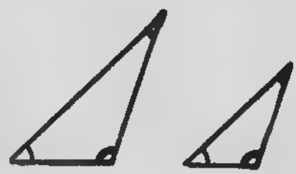
Two triangles are equal in all respects when the following three parts in each are severally equal :

1. *Two sides, and the included angle.* *Theorem* 4.
2. *The three sides.* *Theorem* 7.
3. *Two angles and one side, the side given in one triangle* CORRESPONDING *to that given in the other.* *Theorem* 17.

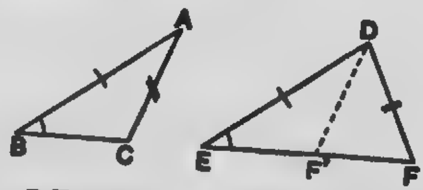
Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other.

For example :

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects.



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* of the other, the given angles being *opposite* to equal sides, the diagram below shews that the triangles need not be equal in all respects.

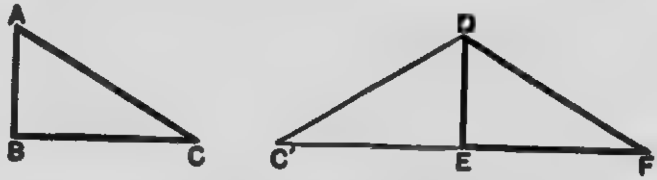


For if $AB = DE$, and $AC = DF$, and the $\angle ABC =$ the $\angle DEF$, it will be seen that the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF'.

NOTE. See also Theorem 18, p. 51, and Problem 9, p. 85.

Theorem 18

Two **right-angled** *triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects.*



Let ABC, DEF be two **right-angled** triangles, in which the $\angle$s ABC, DEF are right angles,
the hypotenuse AC = the hypotenuse DF,
and $AB = DE$.

It is required to prove that the $\triangle$s ABC, DEF *are equal in all respects.*

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$, so that AB falls on the equal line DE, and C on the side of DE opposite to F.

Let C' be the point on which C falls.

Then DEC' represents the $\triangle ABC$ in its new position.

Since each of the $\angle$s DEF, DEC' is a right angle,
$\therefore$ EF and EC' are in one straight line.

And in the $\triangle C'DF$, because $DF = DC'$ (*i.e.* AC),
$\therefore$ the $\angle DFC'$ = the $\angle DC'F$. *Theor.* 5.

Hence in the $\triangle$s DEF, DEC',

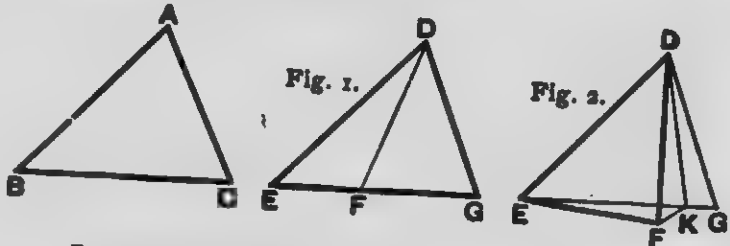
because $\begin{cases} \text{the } \angle DEF = \text{the } \angle DEC', \text{ being right angles ;} \\ \text{the } \angle DFE = \text{the } \angle DC'E, \quad \textit{Proved.} \\ \text{and the side } DE \text{ is common.} \end{cases}$

$\therefore$ the $\triangle$s DEF, DEC' are equal in all respects ; *Theor.* 17.
that is, the $\triangle$s DEF, ABC are equal in all respects.

Q.E.D.

* THEOREM 19. [Euclid I. 24]

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle included by the two sides of one greater than the angle included by the corresponding sides of the other; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two triangles, in which

$BA = ED$, and $AC = DF$,

but the $\angle\ BAC$ is greater than the $\angle\ EDF$.

It is required to prove that BC is greater than EF.

Proof. Apply the $\triangle\ ABC$ to the $\triangle\ DEF$, so that A falls on D, and AB along DE.

Then because $AB = DE$, B must coincide with E.

Let DG, GE represent AC, CB in their new position.

Then if EG passes through F (Fig. 1), EG is greater than EF; that is, BC is greater than EF.

But if EG does not pass through F (Fig. 2), suppose that DK bisects the $\angle\ FDG$, and meets EG in K. Join FK.

Then in the $\triangle$s FDK, GDK,

because $\begin{cases} FD = GD, \text{ and } DK \text{ is common to both,} \\ \text{and the included } \angle\ FDK = \text{the included } \angle\ GDK; \end{cases}$

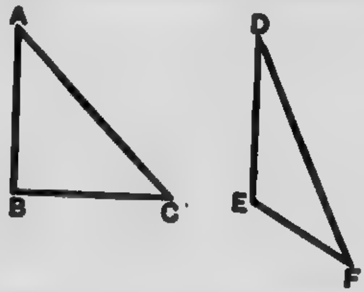
$\therefore\ FK = GK$. *Theor.* 4.

Now the two sides EK, KF are greater than EF;

that is, EK, KG are greater than EF.

$\therefore\ EG$ (or BC) is greater than EF. Q.E.D.

Conversely, *if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base is greater than the angle contained by the corresponding sides of the other.*



Let ABC, DEF be two triangles in which

$BA = ED$,

and $AC = DF$,

but the base BC is greater than the base EF.

It is required to prove that the $\angle\, BAC$ *is greater than the* $\angle\, EDF$.

Proof. If the $\angle\, BAC$ is not greater than the $\angle\, EDF$, it must be either equal to, or less than the $\angle\, EDF$.

Now if the $\angle\, BAC$ were equal to the $\angle\, EDF$, then the base BC would be equal to the base EF; *Theor.* 4.

but, by hypothesis, BC is not equal to EF.

Again, if the $\angle\, BAC$ were less than the $\angle\, EDF$, then the base BC would be less than the base EF; *Theor.* 19.

but, by hypothesis, BC is not less than EF.

That is, the $\angle\, BAC$ is neither equal to, nor less than the $\angle\, EDF$;

$\therefore$ the $\angle\, BAC$ is greater than the $\angle\, EDF$.

Q.E.D.

* *Theorems marked with an asterisk may be omitted or postponed at the discretion of the teacher.*

REVISION LESSON ON TRIANGLES

1. State the properties of a triangle relating to

(i) the sum of its interior angles;
(ii) the sum of its exterior angles.

What property corresponds to (i) in a polygon of n sides? With what other figures does a triangle share the property (ii)?

2. Classify triangles with regard to their angles. Enunciate any Theorem or Corollary assumed in the classification.

3. Enunciate two Theorems in which from data relating to the sides a conclusion is drawn relating to the angles.

In the triangle ABC, if $a = 3{\cdot}6$ cm., $b = 2{\cdot}8$ cm., $c = 3{\cdot}6$ cm., arrange the angles in order of their sizes (before measurement); and prove that the triangle is acute-angled.

4. Enunciate two Theorems in which from data relating to the angles a conclusion is drawn relating to the sides.

In the triangle ABC, if

(i) $A = 48°$ and $B = 51°$, find the third angle, and name the greatest side.

(ii) $A = B = 62\frac{1}{2}°$, find the third angle, and arrange the sides in order of their lengths.

5. From which of the conditions given below may we conclude that the triangles ABC, $A'B'C'$ are identically equal? Point out where ambiguity arises; and draw the triangle ABC in each case.

(i) $\begin{cases} A = A' = 71°. \\ B = B' = 46°. \\ a = a' = 3{\cdot}7 \text{ cm.} \end{cases}$ (ii) $\begin{cases} a = a' = 4{\cdot}2 \text{ cm.} \\ b = b' = 2{\cdot}4 \text{ cm.} \\ C = C' = 81°. \end{cases}$ (iii) $\begin{cases} A = A' = 36°. \\ B = B' = 121°. \\ C = C' = 23°. \end{cases}$

(iv) $\begin{cases} a = a' = 3{\cdot}0 \text{ cm.} \\ b = b' = 5{\cdot}2 \text{ cm.} \\ c = c' = 4{\cdot}5 \text{ cm.} \end{cases}$ (v) $\begin{cases} B = B' = 53°. \\ b = b' = 4{\cdot}3 \text{ cm.} \\ c = c' = 5{\cdot}0 \text{ cm.} \end{cases}$ (vi) $\begin{cases} C = C' = 90°. \\ c = c' = 5 \text{ cm.} \\ a = a' = 3 \text{ cm.} \end{cases}$

6. Summarise the results of the last question by stating generally under what conditions two triangles

(i) are necessarily congruent;
(ii) may or may not be congruent.

7. *If two triangles have their angles equal, each to each, the triangles are not necessarily equal in all respects, because the three data are not* independent. Carefully explain this statement.

(*Miscellaneous Examples*)

8. (i) *The perpendicular is the shortest line that can be drawn to a given straight line from a given point.*

(ii) **Obliques** *which make equal angles with the perpendicular are equal.*

(iii) *Of two obliques the less is that which makes the smaller angle with the perpendicular.*

9. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects.*

10. PQ is a perpendicular (4 cm. in length) to a straight line XY. Draw through P a series of obliques making with PQ the angles 15°, 30°, 45°, 60°, 75°. *Measure* the lengths of these obliques, and tabulate the results.

11. PAB is a triangle in which AB and AP have constant lengths 4 cm. and 3 cm. If AB is fixed, and AP rotates about A, trace the changes in PB, as the angle A increases from 0° to 180°.

Answer this question by drawing a series of figures, increasing A by increments of 30°. Measure PB in each case, and tabulate the results.

12. From B, the foot of a flagstaff AB, a horizontal line is drawn, passing two points C and D which are 27 feet apart. The angles BCA and BDA are 65° and 40° respectively. Represent this on a diagram (scale 1 cm. to 10 ft.), and find by measurement the approximate height of the flagstaff.

13. From P, the top of a lighthouse PQ, two boats A and B are seen at anchor in a line due south of the lighthouse. It is known that $PQ = 126$ ft., $\angle PAQ = 57°$, $\angle PBQ = 33°$; hence draw a plan in which 1″ represents 100 ft., and find by measurement the distance between A and B to the nearest foot.

14. From a lighthouse L two ships A and B, which are 600 yards apart, are observed in directions S.W. and 15° East of South respectively. At the same time B is observed from A in a S.E. direction. Draw a plan (scale 1″ to 200 yds.), and find by measurement the distance of the lighthouse from each ship.

PARALLELOGRAMS

DEFINITIONS

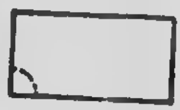
1. A **quadrilateral** is a plane figure bounded by *four* straight lines.

The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

2. A **parallelogram** is a quadrilateral whose opposite sides are *parallel*.

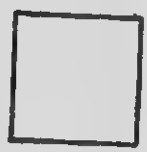
[It will be proved hereafter that the opposite sides of a parallelogram are equal, and that its opposite angles are equal.]

3. A **rectangle** is a parallelogram which has one of its angles a right angle.



[It will be proved hereafter that *all* the angles of a rectangle are right angles. See page 59.]

4. A **square** is a rectangle which has two adjacent sides equal.



[It will be proved that *all* the sides of a square are equal and all its angles right angles. See page 59.]

5. A **rhombus** is a quadrilateral which has all its sides equal, but its angles are not right angles.

6. A **trapezium** is a quadrilateral which has *one* pair of parallel sides.

THEOREM 20. [Euclid I. 33]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD.

It is required to prove that AC and BD are equal and parallel.

Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

$\therefore$ the $\angle ABC$ = the alternate $\angle DCB$.

Now in the $\triangle$ ABC, DCB,

because $\begin{cases} AB = DC, \\ BC \text{ is common to both;} \\ \text{and the } \angle ABC = \text{the } \angle DCB; \end{cases}$ *Proved.*

$\therefore$ the triangles are equal in all respects;

so that $AC = DB$,..................(i)

and the $\angle ACB = \angle DBC$.

But these are alternate angles;

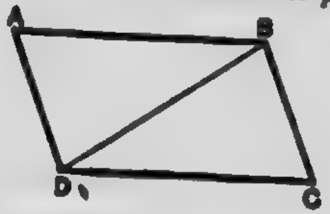
$\therefore$ AC and BD are parallel. ....(ii)

That is, AC and BD are both equal and parallel.

Q.E.D.

Theorem 21. [Euclid I. 34]

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let $ABCD$ be a parallelogram, of which BD is a diagonal.

It is required to prove that

(i) $AB = CD$, *and* $AD = CB$,
(ii) *the* $\angle BAD =$ *the* $\angle DCB$,
(iii) *the* $\angle ADC =$ *the* $\angle CBA$,
(iv) *the* $\triangle ABD =$ *the* $\triangle CDB$ *in area.*

Proof. Because AB and DC are parallel, and BD meets them,

$\therefore$ the $\angle ABD =$ the alternate $\angle CDB$.

Because AD and BC are parallel, and BD meets them,

$\therefore$ the $\angle ADB =$ the alternate $\angle CBD$.

Hence in the $\triangle$s ABD, CDB,

$$\text{because}\left\{\begin{array}{l}\text{the } \angle ABD = \text{the } \angle CDB,\\ \text{the } \angle ADB = \text{the } \angle CBD,\\ \text{and } BD \text{ is common to both ;}\end{array}\right. \qquad \textit{Proved.}$$

$\therefore$ the triangles are equal in all respects ; *Theor.* 17.

so that $AB = CD$, and $AD = CB$;..............(i)

and the $\angle BAD =$ the $\angle DCB$; ....(ii)

and the $\triangle ABD =$ the $\triangle CDB$ in area. ....(iv)

And because the $\angle ADB =$ the $\angle CBD$, *Proved.*

and the $\angle CDB =$ the $\angle ABD$,

$\therefore$ the whole $\angle ADC =$ the whole $\angle CBA$. . (iii)

Q.E.D.

Corollary 1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

In other words:

All *the angles of a rectangle are right angles.*

For the sum of two consecutive $\angle s = 2$ rt. $\angle s$; (*Theor.* 14.)

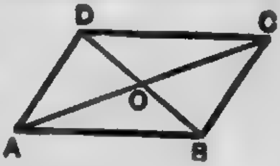
$\therefore$, if one of these is a rt. angle, the other must be a rt. angle.

And the opposite angles of the par^m are equal;

$\therefore$ all the angles are right angles.

Corollary 2. *All the sides of a square are equal; and all its angles are right angles.*

Corollary 3. *The diagonals of a parallelogram bisect one another.*



Let the diagonals AC, BD of the par^m $ABCD$ intersect at O.

To prove $AO = OC$, *and* $BO = OD$.

In the $\triangle s$ AOB, COD,

because $\begin{cases} \text{the } \angle OAB = \text{the alt. } \angle OCD, \\ \text{the } \angle AOB = \text{vert. opp. } \angle COD, \\ \quad\text{and } AB = \text{the opp. side } CD; \end{cases}$

$\therefore OA = OC$; and $OB = OD$. *Theor.* 17.

EXERCISES

1. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

2. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

3. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

4. *The diagonals of a rhombus bisect one another at right angles.*

5. If the diagonals of a parallelogram are equal, all its angles are right angles.

6. In a parallelogram which is not rectangular the diagonals are unequal.

EXERCISES ON PARALLELS AND PARALLELOGRAMS

(Symmetry and Superposition)

1. Shew that by folding a rhombus about one of its diagonals the triangles on opposite sides of the crease may be made to coincide.

That is to say, prove that a rhombus is *symmetrical* about either diagonal.

2. Prove that the diagonals of a square are *axes of symmetry.* Name two other lines about which a square is symmetrical.

3. The diagonals of a rectangle divide the figure into two congruent triangles: is the diagonal, therefore, an axis of symmetry? About what two lines is a rectangle symmetrical?

4. Is there any axis about which an oblique parallelogram is symmetrical? Give reasons for your answer.

5. In a quadrilateral $ABCD$, $AB = AD$ and $CB = CD$; but the sides are not all equal. Which of the diagonals (if either) is an axis of symmetry?

6. Prove by the method of superposition that

(i) *Two parallelograms are identically equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other.*

(ii) *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

7. Two quadrilaterals $ABCD$, $EFGH$ have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH. Shew that the figures may be made to coincide with one another.

(Miscellaneous Theoretical Examples)

8. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

9. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

10. If $ABCD$ is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure $AYCX$ is a parallelogram.

11. ABC and DEF are two triangles such that AB, BC are respectively equal to and parallel to DE, EF; shew that AC is equal and parallel to DF.

12. $ABCD$ is a quadrilateral in which AB is parallel to DC, and AD equal but not parallel to BC; shew that

(i) the $\angle A +$ the $\angle C = 180° =$ the $\angle B +$ the $\angle D$;

(ii) the diagonal $AC =$ the diagonal BD;

(iii) the quadrilateral is *symmetrical* about the straight line joining the middle points of AB and DC.

13. AP, BQ are straight rods of equal length, turning at equal rates (both clockwise) about two fixed pivots A and B respectively. If the rods start parallel but pointing in opposite senses, shew that

(i) they will always be parallel;

(ii) the line joining PQ will always pass through a fixed point.

(Miscellaneous Numerical and Graphical Examples)

14. A yacht sailing due East changes her course successively by 63°, by 78°, by 119°, and by 64°, with a view to sailing round an island. What further change must be made to set her once more on an Easterly course?

15. If the sum of the interior angles of a rectilineal figure is equal to the sum of the exterior angles, how many sides has it, and why?

16. Draw, using your protractor, any five-sided figure $ABCDE$, in which

$$\angle B = 110°, \quad \angle C = 115°, \quad \angle D = 93°, \quad \angle E = 152°.$$

Verify by a construction with ruler and compasses that AE is parallel to BC, and account theoretically for this fact.

17. A and B are two fixed points, and two straight lines AP, BQ, unlimited towards P and Q, are pivoted at A and B. AP, starting from the direction AB, turns about A clockwise at the uniform rate of $7\frac{1}{2}°$ a second; and BQ, starting simultaneously from the direction BA, turns about B counter-clockwise at the rate of $3\frac{1}{2}°$ a second.

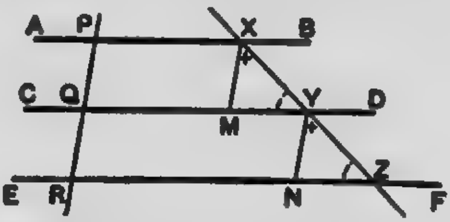
(i) In how many seconds will AP and BQ be parallel?

(ii) Find graphically and by calculation the angle between AP and BQ twelve seconds from the start.

(iii) At what rate does this angle decrease?

THEOREM 22

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal.



Let the parallels AB, CD, EF cut off equal intercepts PQ, QR from the transversal PQR; and let XY, YZ be the corresponding intercepts cut off from any other transversal XYZ.

It is required to prove that $XY = YZ$.

Through X and Y let XM and YN be drawn parallel to PR.

Proof. Since CD and EF are parallel, and XZ meets them,

$\therefore$ the $\angle XYM$ = the corresponding $\angle YZN$.

And since XM, YN are parallel, each being parallel to PR,

$\therefore$ the $\angle MXY$ = the corresponding $\angle NYZ$.

Now the figures PM, QN are parallelograms,

$\therefore$ XM = the opp. side PQ, and YN = the opp. side QR;

and since by hypothesis $PQ = QR$,

$\therefore XM = YN$.

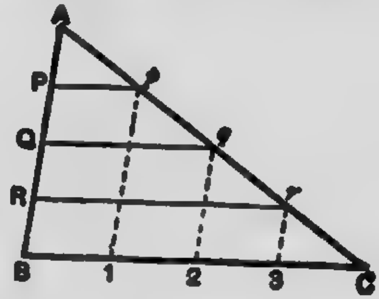
Then in the $\triangle$ XMY, YNZ,

because $\begin{cases} \text{the } \angle XYM = \text{the } \angle YZN, \\ \text{the } \angle MXY = \text{the } \angle NYZ, \\ \text{and } XM = YN; \end{cases}$

$\therefore$ the triangles are identically equal; *Theor.* 17.

$\therefore XY = YZ$. Q.E.D.

Corollary. *In a triangle ABC, if a set of lines Pp, Qq, Rr, . . . , drawn parallel to the base, divide one side AB into equal parts, they also divide the other side AC into equal parts.*



Note. The lengths of the parallels Pp, Qq, Rr, . . . , may thus be expressed in terms of the base BC.

Through p, q, and r let $p1$, $q2$, $r3$ be drawn par[l] to AB.

Then, by Theorem 22, these par[ls] divide BC into *four* equal parts, of which Pp evidently contains *one*, Qq *two*, and Rr *three*.

In other words,

$$Pp = \tfrac{1}{4} \cdot BC; \quad Qq = \tfrac{2}{4} \cdot BC; \quad Rr = \tfrac{3}{4} \cdot BC.$$

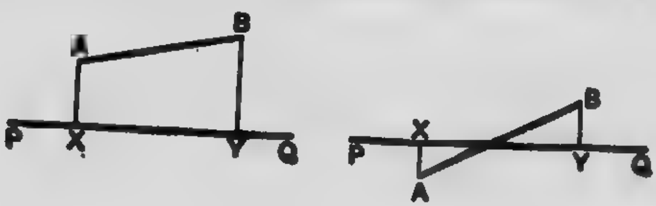
Similarly if the given par[ls] divide AB into n equal parts,

$$Pp = \frac{1}{n} \cdot BC, \quad Qq = \frac{2}{n} \cdot BC, \quad Rr = \frac{3}{n} \cdot BC; \text{ and so on.}$$

*** *Problem 7, p. 78, should now be worked.*

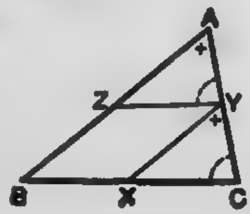
DEFINITION

If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the **orthogonal projection** of AB on PQ.



EXERCISES ON PARALLELS AND PARALLELOGRAMS

1. *The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side.*

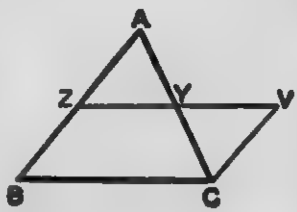


[This is an important particular case of Theorem 22.

In the $\triangle ABC$, if Z is the middle point of AB, and ZY is drawn par[l] to BC, we have to prove that $AY = YC$.

Draw YX par[l] to AB, and then prove the ▲ ZAY, XYC congruent.]

2. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*



[In the $\triangle ABC$, if Z, Y are the middle points of AB, AC, we have to prove ZY par[l] to BC.

Produce ZY to V, making YV equal to ZY, and join CV. Prove the ▲ AYZ, CYV congruent.]

3. *The straight line which joins the middle points of two sides of a triangle is equal to half the third side.*

4. *Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four congruent triangles.*

5. *Any straight line drawn from the vertex of a triangle to the base is bisected by the line which joins the middle points of the other sides.*

6. $ABCD$ is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect AC.

7. *If the middle points of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram.*

8. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

9. From two points A and B, and from O the mid-point between them, perpendiculars AP, BQ, OX are drawn to a straight line CD. If AP, BQ measure respectively 4·2 cm. and 5·8 cm., *deduce* the length of OX, and verify your result by measurement.

Shew that $OX = \frac{1}{2}(AP + BQ)$ or $\frac{1}{2}(AP - BQ)$, according as A and B are on the *same* side, or on *opposite* sides of CD.

10. When three parallel lines cut off equal intercepts from two transversals, shew that of the three parallel lengths between the two transversals the middle one is the **Arithmetic Mean** of the other two.

11. *The parallel sides of a trapezium are* a *centimetres and* b *centimetres in length. Prove that the line joining the middle points of the oblique sides is parallel to the parallel sides, and that its length is* $\frac{1}{2}(a + b)$ *centimetres.*

12. OX and OY are two straight lines, and along OX five points 1, 2, 3, 4, 5 are marked at equal distances. Through these points parallels are drawn in any direction to meet OY. Measure the lengths of these parallels: take their average, and compare it with the length of the *third* parallel. Prove *geometrically* that the 3rd parallel is the mean of all five.

State the corresponding theorem for any odd number $(2n + 1)$ of parallels so drawn.

13. From the angular points of a parallelogram perpendiculars are drawn to any straight line which is outside the parallelogram: shew that the sum of the perpendiculars drawn from one pair of opposite angular points is equal to the sum of those drawn from the other pair.

[Draw the diagonals, and from their point of intersection suppose a perpendicular drawn to the given straight line.]

14. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the equal sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

[It follows that the sum of the distances of *any* point in the base of an isosceles triangle from the equal sides is **constant**, that is, the same whatever point in the base is taken.]

How would this property be modified if the given point were taken in the base *produced?*

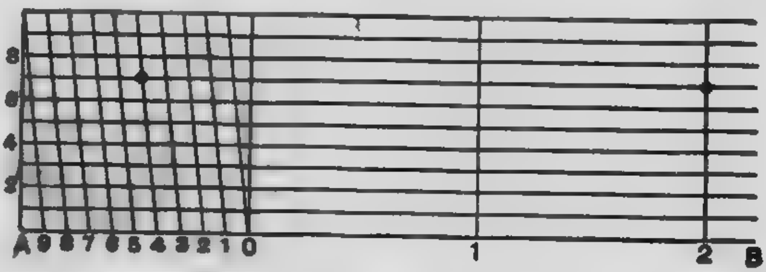
15. The sum of the perpendiculars drawn from any point within an equilateral triangle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant.

16. Equal and parallel lines have equal projections on any other straight line.

DIAGONAL SCALES

Diagonal scales form an important application of Theorem 22. We shall illustrate their construction and use by describing a *Decimal Diagonal Scale to shew Inches, Tenths and Hundredths.*

A straight line AB is divided (from A) into inches, and the points of division marked **0**, **1**, **2**, ... The primary division **0** A is subdivided into *tenths*, these secondary divisions being numbered (from **0**) **1**, **2**, **3**, ... **9**. We may now read on AB *inches* and *tenths* of an inch.



In order to read *hundredths*, ten lines are taken at any equal, intervals parallel to AB; and perpendiculars are drawn through **0**, **1**, **2**, ...

The primary (or inch) division corresponding to **0** A on the tenth parallel is now subdivided into *ten* equal parts; and diagonal lines are drawn, as in the diagram,

joining **0** to the *first* point of subdivision on the 10th parallel,
" **1** to the *second* " " " " ,
" **2** to the *third* " " " " ;

and so on.

The scale is now complete, and its use is shewn in the following example.

Example. To take from the scale a length of 2·47 *inches.*

(i) Place one point of the dividers at **2** in AB, and extend them

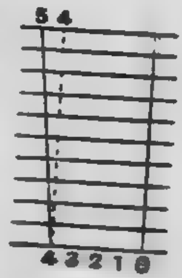
till the other point reaches 4 in the subdivided inch 0 A. We have now 2·4 inches in the dividers.

(ii) To get the remaining 7 *hundredths*, move the right-hand point up the perpendicular through 2 till it reaches the 7th parallel. Then extend the dividers till the left point reaches the diagonal 4 also on the 7th parallel. We have now 2·47 inches in the dividers.

REASON FOR THE ABOVE PROCESS

The first step needs no explanation. The reason of the second is found in the Corollary of Theorem 22.

Joining the point 4 to the corresponding point on the tenth parallel, we have a triangle **4,4,5**, of which one side **4,4** is divided into ten equal parts by lines parallel to **4,5**.



Therefore the lengths of the parallels between **4,4**, and the diagonal **4,5** are $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, ... of the base, which is ·1 inch.

Hence these lengths are ·01, ·02, ·03, ... of 1 inch.

Thus, by means of the scale, the length of a straight line may be measured to the nearest hundredth of an inch.

Again, if one inch-division on the scale is taken to represent 10 feet, then 2·47 inches on the scale will represent 24·7 feet. And if one inch-division on the scale represents 100 links, then 2·47 inches will represent 247 links. Thus a diagonal scale is of service in preparing plans of enclosures, buildings, or field-works, where it is necessary that every dimension of the actual object must be represented by a line of proportional length on the plan.

NOTE

The subdivision of a diagonal scale need not be *decimal*.

For instance we might construct a diagonal scale to read centimetres, millimetres, and *quarters* of a millimetre; in which case we should take *four* parallels to the line AB.

EXERCISES ON LINEAR MEASUREMENTS

1. Draw straight lines whose lengths are 1·25 inches, 2·72 inches, 3·08 inches.

2. Draw a line 2·68 inches long, and measure its length in centimetres and the nearest millimetre.

3. Draw a line 5·7 cm. in length, and measure it in inches (to the nearest hundredth). Check your result by calculation, given that 1 cm. = 0·3937 inch.

4. Find by measurement the equivalent of 3·15 inches in centimetres and millimetres. Hence calculate (correct to two decimal places) the value of 1 cm. in inches.

5. Draw lines 2·9 cm. and 6·2 cm. in length, and measure them in inches. Use each equivalent, to find the value of 1 inch in centimetres and millimetres, and take the average of your results.

6. A distance of 100 miles is represented on a map by 1 inch. Draw lines to represent distances of 336 miles and 408 miles.

7. If 1 inch on a map represents 1 kilometre, draw lines to represent 850 metres, 2980 metres, and 1010 metres.

8. A plan is drawn to the scale of 1 inch to 100 links. Measure in centimetres and millimetres a line representing 417 links.

9. Find to the nearest hundredth of an inch the length of a line which will represent 42·500 kilometres in a map drawn to the scale of 1 centimetre to 5 kilometres.

10. The distance from London to Oxford (in a direct line) is 55 miles. If this distance is represented on a map by 2·75 inches, to what scale is the map drawn? That is, how many miles will be represented by 1 inch? How many kilometres by 1 centimetre?

[1 cm. = 0·3937 inch; 1 km. = $\frac{5}{8}$ mile, nearly.]

11. On a map of France drawn to the scale 1 inch to 35 miles, the distance from Paris to Calais is represented by 4·2 inches. Find the distance accurately in miles, and approximately in kilometres, and express the scale in metric measure. [1 km. = $\frac{5}{8}$ mile, nearly.]

12. The distance from Exeter to Plymouth is 37½ miles, and appears on a certain map to be 2½″; and the distance from Lincoln to York is 88 km., and appears on another map to be 7 cm. Compare the scales of these maps in miles to the inch.

13. Draw a diagonal scale, 2 centimetres to represent 1 yard, shewing yards, feet, and inches.

PRACTICAL GEOMETRY

PROBLEMS

The following problems are to be solved with ruler and compasses only. No step requires the actual measurement of any line or angle ; that is to say, the *constructions* are to be made without using either a graduated scale of length, or a protractor.

The problems are not merely to be studied as propositions ; but the construction in every case is to be actually performed by the learner, great care being given to accuracy of drawing.

Each problem is followed by a *theoretical* proof ; but the *results* of the work should always be verified by measurement, as a test of correct drawing. Accurate measurement is also required in applications of the problems.

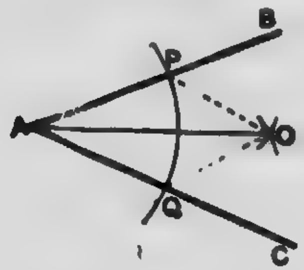
In the diagrams of the problems lines which are inserted only for purposes of *proof* are dotted, to distinguish them from lines necessary to the construction.

For practical applications of the problems the student should be provided with the following instruments :

1. A flat ruler, one edge being graduated in centimetres and millimetres, and the other in inches and tenths.
2. Two set squares ; one with angles of 45°, and the other with angles of 60° and 30°.
3. A pair of pencil compasses.
4. A pair of dividers, preferably with screw adjustment.
5. A semi-circular protractor.

Problem 1

To bisect a given angle.



Let BAC be the given angle to be bisected.

Construction. With centre A, and any radius, draw an arc of a circle cutting AB, AC at P and Q.

With centres P and Q, and radius PQ draw two arcs cutting at O. Join AO.

Then the $\angle BAC$ is bisected by AO.

Proof. Join PO, QO.

In the $\triangle$ APO, AQO,

because $\left\{\begin{array}{l} AP = AQ, \text{ being radii of a circle,} \\ PO = QO, \quad \text{“} \quad \text{“} \quad \text{equal circles,} \\ \text{and } AO \text{ is common;} \end{array}\right.$

$\therefore$ the triangles are equal in all respects; *Theor.* 7.

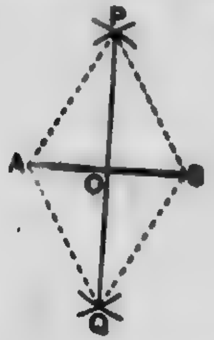
so that the $\angle PAO$ = the $\angle QAO$;

that is, the $\angle BAC$ is bisected by AO.

Note. PQ has been taken as the radius of the arcs drawn from the centres P and Q, and the intersection of these arcs determines the point O. *Any* radius, however, may be used instead of PQ, provided that it is great enough to secure the intersection of the arcs.

PROBLEM 2

To bisect a given straight line.



Let AB be the line to be bisected.

Construction. With centre A, and radius AB, draw two arcs, one on each side of AB.

With centre B, and radius BA, draw two arcs, one on each side of AB, cutting the first arcs at P and Q.

Join PQ, cutting AB at O.

Then AB is bisected at O.

Proof. Join AP, AQ, BP, BQ.

In the ▲ APQ, BPQ,

because $\begin{cases} AP = BP, \text{ being radii of equal circles,} \\ AQ = BQ, \text{ for the same reason,} \\ \text{and } PQ \text{ is common;} \end{cases}$

$\therefore$ the $\angle APQ =$ the $\angle BPQ$. *Theor.* 7.

Again in the ▲ APO, BPO,

because $\begin{cases} AP = BP, \text{ and } PO \text{ is common,} \\ \text{and the } \angle APO = \text{the } \angle BPO; \end{cases}$

$\therefore AO = OB$; *Theor.* 4.

that is, AB is bisected at O.

NOTE. From the congruence of the ▲ APO, BPO it follows that the $\angle AOP =$ the $\angle BOP$. As these are adjacent angles, it follows that *PQ bisects AB at right angles.*

Problem 3

To draw a straight line perpendicular to a given straight line at a given point in it.

Let AB be the straight line, and X the point in it at which a perpendicular is to be drawn.

Construction. With centre X cut off from AB any two equal parts XP, XQ.

With centres P and Q, and radius PQ, draw two arcs cutting at O.

Join XO.

Then XO is perp. to AB.

Proof. Join OP, OQ.

In the $\triangle$ OXP, OXQ,

because $\begin{cases} XP = XQ, \text{ by construction,} \\ OX \text{ is common,} \\ \text{and } PO = QO, \text{ being radii of equal circles;} \end{cases}$

$\therefore$ the $\angle\, OXP =$ the $\angle\, OXQ$. *Theor.* 7.

And these being adjacent angles, each is a right angle;

that is, XO is perp. to AB.

Obs. If the point X is near one end of AB, one or other of the alternative constructions on the next page should be used.

Problem 3. Second Method

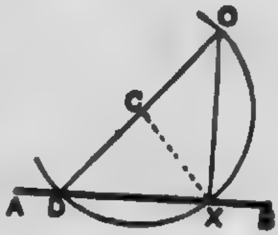
Construction. Take any point C outside AB.

With centre C, and radius CX, draw a circle cutting AB at D.

Join DC, and produce it to meet the circumference of the circle at O.

Join XO.

Then XO is perp. to AB.



Proof. Join CX.

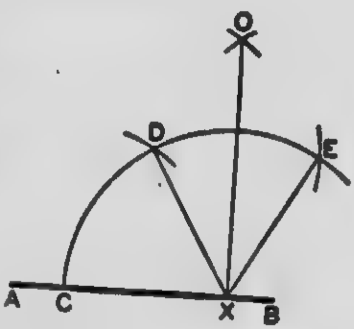
Because $CO = CX$; $\therefore$ the $\angle CXO =$ the $\angle COX$;
and because $CD = CX$; $\therefore$ the $\angle CXD = \angle CDX$.
$\therefore$ the whole $\angle DXO =$ the $\angle XOD +$ the $\angle XDO$
$= \frac{1}{2}$ of $180° = 90°$.
$\therefore$ XO is perp. to AB.

Problem 3. Third Method

Construction. With centre X and any radius, draw the arc CDE, cutting AB at C.

With centre C, and with the same radius, draw an arc, cutting the first arc at D.

With centre D, and with the same radius, draw an arc, cutting the first arc at E.

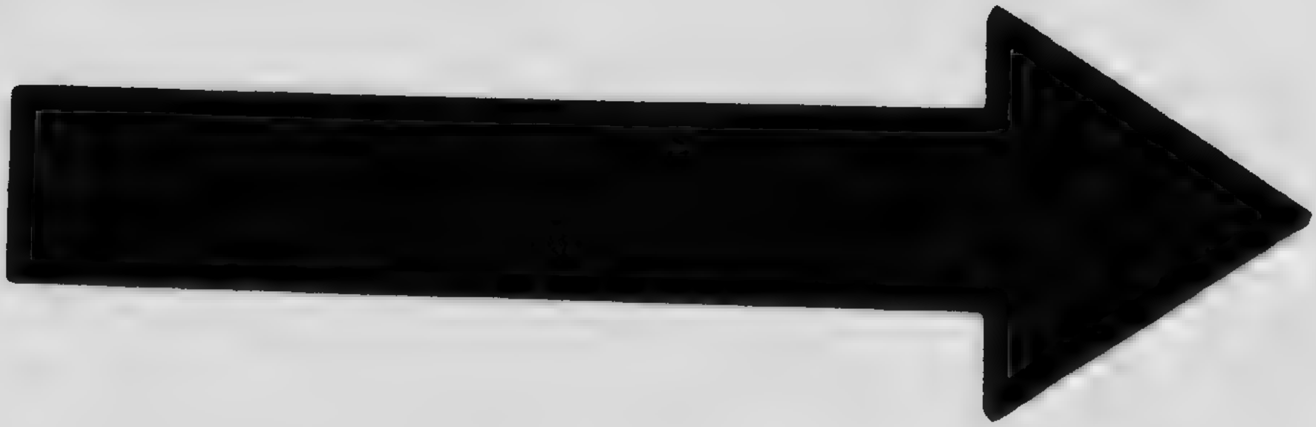


Bisect the $\angle DXE$ by XO.

Then XO is perp. to AB.

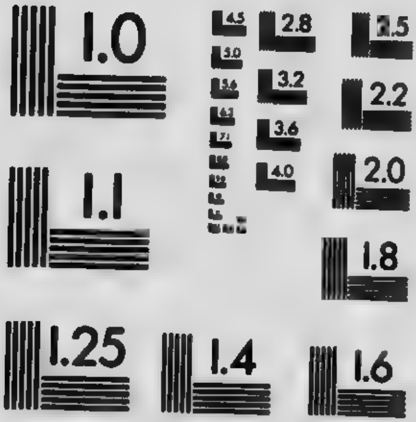
Prob. 1.

Proof. Each of the $\angle$s CXD, DXE is $60°$;
and the $\angle DXO$ is half of the $\angle DXE$;
$\therefore$ the $\angle CXO$ is $90°$.
That is, XO is perp. to AB.



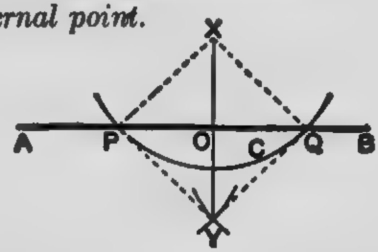
MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



PROBLEM 4

To draw a straight line perpendicular to a given straight line from a given external point.



Let X be the given external point from which a perpendicular is to be drawn to AB.

Construction. Take any point C on the side of AB remote from X.

With centre X, and radius XC, draw an arc to cut AB at P and Q.

With centres P and Q, and radius PX, draw arcs cutting at Y, on the side of AB opposite to X.

Join XY cutting AB at O.

Then XO is perp. to AB.

Proof. Join PX, QX, PY, QY.

In the $\triangle$ PXY, QXY,

because $\begin{cases} PX = QX, \text{ being radii of a circle,} \\ PY = QY, \text{ for the same reason,} \\ \text{and } XY \text{ is common;} \end{cases}$

$\therefore$ the $\angle\, PXY =$ the $\angle\, QXY$. *Theor.* 7.

Again, in the $\triangle$ PXO, QXO,

because $\begin{cases} PX = QX, \\ XO \text{ is common,} \\ \text{and the } \angle\, PXO = \text{the } \angle\, QXO; \end{cases}$

$\therefore$ the $\angle\, XOP =$ the $\angle\, XOQ$. *Theor.* 4.

And these being adjacent angles, each is a right angle,

that is, XO is perp. to AB.

Obs. When the point X is nearly opposite one end of AB, one or other of the alternative constructions given below should be used.

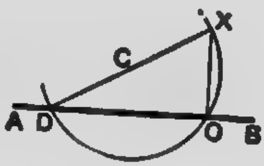
Problem 4. Second Method

Construction. Take any point D in AB. Join DX, and bisect it at C.

With centre C, and radius CX, draw a circle cutting AB at D and O.

Join XO.

Then XO is perp. to AB.



For, as in Problem 3, Second Method, the $\angle XOD$ is a right angle.

Problem 4. Third Method

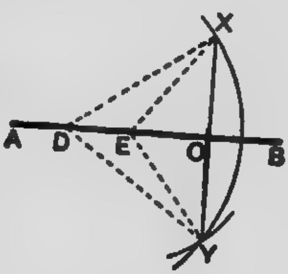
Construction. Take any two points D and E in AB.

With centre D, and radius DX, draw an arc of a circle, on the side of AB opposite to X.

With centre E, and radius EX, draw another arc cutting the former at Y.

Join XY, cutting AB at O.

Then XO is perp. to AB.



(i) Prove the ▲ XDE, YDE equal in all respects by Theorem 7,

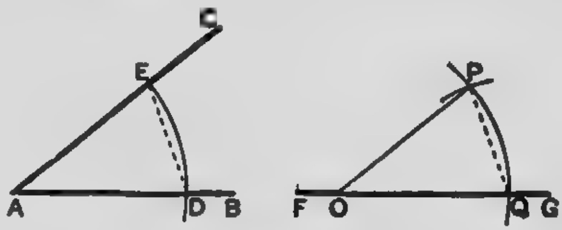
so that the $\angle XDE$ = the $\angle YDE$.

(ii) Hence prove the ▲ XDO, YDO equal in all respects by Theorem 4, so that the adjacent ∠s DOX, DOY are equal.

That is, XO is perp. to AB.

Problem 5

At a given point in a given straight line to make an angle equal to a given angle.



Let BAC be the given angle, and FG the given straight line; and let O be the point at which an angle is to be made equal to the $\angle\, BAC$.

Construction. With centre A, and with any radius, draw an arc cutting AB and AC at D and E.

With centre O, and with the same radius, draw an arc cutting FG at Q.

With centre Q, and with radius DE, draw an arc cutting the former arc at P.

Join OP.

Then POQ is the required angle.

Proof. Join ED, PQ.

In the $\triangle$ POQ, EAD,

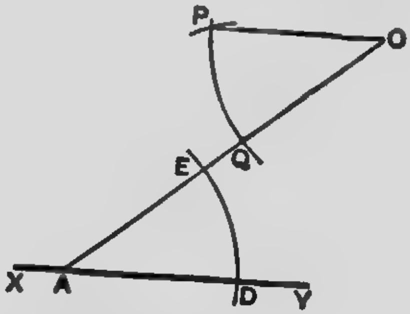
$$\text{because}\begin{cases} OP = AE, \text{ being radii of equal circles,} \\ OQ = AD, \text{ for the same reason,} \\ PQ = ED, \text{ by construction;} \end{cases}$$

$\therefore$ the triangles are equal in all respects;

so that the $\angle\, POQ =$ the $\angle\, EAD$. *Theor.* 7.

Problem 6

Through a given point to draw a straight line parallel to a given straight line.



Let XY be the given straight line, and O the given point, through which a straight line is to be drawn par^l to XY.

Construction. In XY take any point A, and join OA.

Using the construction of Problem 5, at the point O on the line AO make the $\angle\, AOP$ equal to the $\angle\, OAY$ and alternate to it.

Then OP is parallel to XY.

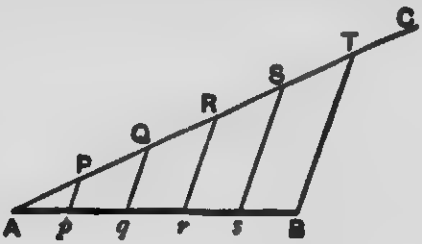
Proof. Because AO, meeting the straight lines OP, XY, makes the alternate $\angle s$ POA, OAY equal;

$\therefore$ OP is par^l to XY.

*** *The constructions of Problems* 3, 4, *and* 6 *are not usually followed in practical applications. Parallels and perpendiculars may be more quickly drawn by the aid of set squares.* (*See* Lessons in Experimental Geometry, *pp.* 36, 42.)

PROBLEM 7

To divide a given straight line into any number of equal parts.



Let AB be the given straight line, and suppose it is required to divide it into *five* equal parts.

Construction. From A draw AC, a straight line of unlimited length, making any angle with AB.

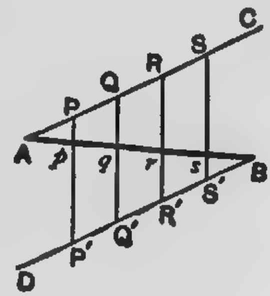
From AC mark off *five* equal parts of *any* length, AP, PQ, QR, RS, ST.

Join TB; and through P, Q, R, S draw par[ls] to TB, meeting AB in p, q, r, s.

Then since the par[ls] Pp, Qq, Rr, Ss, TB cut off five equal parts from AT, they also cut off five equal parts from AB. (Theorem 22.)

SECOND METHOD

From A draw AC at any angle with AB, and on it mark off *four* equal parts AP, PQ, QR, RS, of any length.



From B draw BD par[l] to AC, and on it mark off BS', $S'R'$, $R'Q'$, $Q'P'$, each equal to the parts marked on AC.

Join PP', QQ', RR', SS' meeting AB in p, q, r, s. Then AB is divided into five equal parts at these points.

[Prove by Theorems 20 and 22.]

EXERCISES ON LINES AND ANGLES

(Graphical Exercises)

1. Construct (with ruler and compasses only) an angle of 60°. By repeated bisection divide this angle into *four* equal parts.

2. By means of Exercise 1, *trisect a right angle*; that is, divide it into *three* equal parts.

Bisect each part, and hence shew how to trisect an angle of 45°. [No construction is known for exactly trisecting *any* angle.]

3. Draw a line 6·7 cm. long, and divide it into *five* equal parts. Measure one of the parts in inches (to the nearest hundredth), and verify your work by calculation. [1 cm. = 0·3937 inch.]

4. From a straight line 3·72″ long, cut off *one seventh*. Measure the part in centimetres and the nearest millimetre, and verify your work by calculation.

5. At a point X in a straight line AB draw XP perpendicular to AB, making XP 1·8″ in length. From P draw an oblique PQ, 3·0″ long, to meet AB in Q. Measure XQ.

(Problems. State your construction, and give a theoretical proof)

6. In a straight line XY find a point which is equidistant from two given points A and B.

When is this impossible?

7. In a straight line XY find a point which is equidistant from two intersecting lines AB, AC.

When is this impossible?

8. From a given point P draw a straight line PQ, making with a given straight line AB an angle of given magnitude.

9. From two given points P and Q on the same side of a straight line AB, draw two lines which meet in AB and make equal angles with it.

[*Construction.* From P draw PH perp. to AB, and produce PH to P', making HP' equal to PH. Join $P'Q$ cutting AB at K. Join PK. Prove that PK, QK are the required lines.]

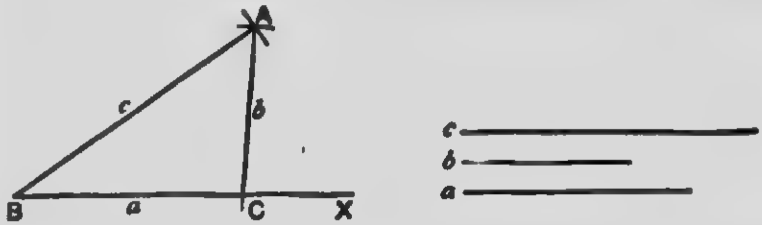
10. Through a given point P draw a straight line such that the perpendiculars drawn to it from two points A and B may be equal.

Is this always possible?

THE CONSTRUCTION OF TRIANGLES

Problem 8

To draw a triangle having given the lengths of the three sides.



Let a, b, c be the lengths to which the sides of the required triangle are to be equal.

Construction. Draw any straight line BX, and cut off from it a part BC equal to a.

With centre B, and radius c, draw an arc of a circle.

With centre C, and radius b, draw a second arc cutting the first at A.

Join AB, AC.

Then ABC is the required triangle, for by construction the sides BC, CA, AB are equal to a, b, c respectively.

Obs. The three data a, b, c may be understood in two ways: either as three actual lines to which the sides of the triangle are to be equal, or as three *numbers* expressing the lengths of those lines in terms of inches, centimetres, or some other linear unit.

Notes. (i) In order that the construction ma be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 11); for otherwise the arcs drawn from the centres B and C would not cut.

(ii) The arcs which cut at A would, if continued, cut again on the other side of BC. Thus the construction gives two triangles on opposite sides of a common base.

ON THE CONSTRUCTION OF TRIANGLES

It has been seen (page 50) that to prove two triangles identically equal, *three* parts of one must be given equal to the corresponding parts of the other (though *any* three parts do not necessarily serve the purpose). This amounts to saying that *to determine the shape and size of a triangle we must know* **three** *of its parts:* or, in other words,

To construct a triangle three independent data are required.

For example, we may construct a triangle

(i) When *two sides* (b, c) and the *included angle* (A) are given.

The method of construction in this case is obvious.

(ii) When *two angles* (A, B) and one side (a) are given.

Here, since A and B are given, we at once know C;

for $A + B + C = 180°$.

Hence we have only to draw the base equal to a, and at its ends make angles equal to B and C; for we know that the remaining angle must necessarily be equal to A:

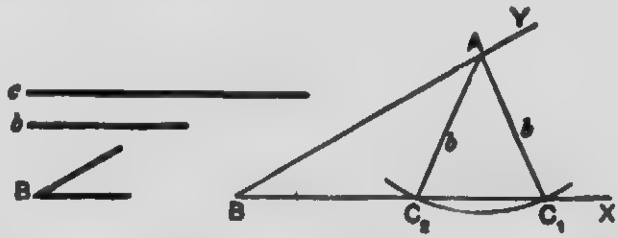
(iii) If the *three angles* A, B, C are given (and no side), the problem is **indeterminate**, that is, the number of solutions is unlimited.

For if at the ends of *any* base we make angles equal to B and C, the third angle is equal to A.

This construction is indeterminate, because the three data are not *independent*, the third following necessarily from the other two.

PROBLEM 9

To construct a triangle having given two sides and an angle opposite to one of them.



Let b, c be the given sides and B the given angle.

Construction. Take any straight line BX, and at B make the $\angle XBY$ equal to the given $\angle B$.

From BY cut off BA equal to c.

With centre A, and radius b, draw an arc of a circle.

If this arc cuts BX in two points C_1 and C_2, both on the same side of B, both of the $\triangle ABC_1$, ABC_2 satisfy the given conditions.

This double solution is known as the **Ambiguous Case**, and will occur when b is less than c but greater than the perp. from A on BX.

EXERCISE

Draw figures to illustrate the nature and number of solutions in the following cases:

(i) When b is greater than c.

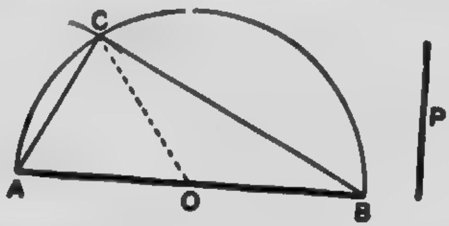
(ii) When b is equal to c.

(iii) When b is equal to the perpendicular from A on BX.

(iv) When b is less than this perpendicular.

Problem 10

To construct a right-angled triangle having given the hypotenuse and one side.



Let AB be the hypotenuse and P the given side.

Construction. Bisect AB at O; and with centre O, and radius OA, draw a semicircle.

With centre A, and radius P, draw an arc to cut the semicircle at C.

Join AC, BC.

Then ABC is the required triangle.

Proof. Join OC.

Because $OA = OC$;

$\therefore$ the $\angle OCA =$ the $\angle OAC$.

And because $OB = OC$;

$\therefore$ the $\angle OCB =$ the $\angle OBC$.

$\therefore$ the whole $\angle ACB =$ the $\angle OAC +$ the $\angle OBC$

$= \frac{1}{2}$ of $180°$ *Theor.* 16.

$= 90°$.

ON THE CONSTRUCTION OF TRIANGLES

(Graphical Exercises)

1. Draw a triangle whose sides are 7·5 cm., 6·2 cm., and 5·3 cm. Draw and measure the perpendiculars dropped on these sides from the opposite vertices.

2. Draw a triangle, given $a = 3{\cdot}00''$, $b = 2{\cdot}50''$, $c = 2{\cdot}75''$. Bisect the angle A by a line which meets the base at X. Measure BX and XC (to the nearest hundredth of an inch); and hence calculate the value of $\frac{BX}{CX}$ to two places of decimals. Compare your result with the value of c/b.

3. Two sides of triangular field are 315 yards and 260 yards, and the included angle is known to be 39°. Draw a plan (1 inch to 100 yards) and find by measurement the length of the remaining side of the field.

4. ABC is a triangular plot of ground, of which the base BC is 75 metres, and the angles at B and C are 47° and 68° respectively. Draw a plan (scale 1 cm. to 10 metres). Write down without measurement the size of the angle A; and by measuring the plan, obtain the approximate lengths of the other sides of the field; also the perpendicular drawn from A to BC.

5. A yacht on leaving harbour steers N.E. sailing 9 knots an hour. After 20 minutes she goes about, steering N.W. for 35 minutes and making the same average speed as before. How far is she now from the harbour, and what course (approximately) must she set for the run home? Obtain your results from a chart of the whole course, scale 2 cm. to 1 knot.

6. Draw a right-angled triangle, given that the hypotenuse $c = 10{\cdot}6$ cm. and one side $a = 5{\cdot}6$ cm. Measure the third side b; and find the value of $\sqrt{c^2-a^2}$. Compare the two results.

7. Construct a triangle, having given the following parts: $B = 34°$, $b = 5{\cdot}5$ cm., $c = 8{\cdot}5$ cm. Shew that there are two solutions. Measure the two values of a, and also of C, and shew that the latter are supplementary.

8. In a triangle ABC, the angle $A = 50°$, and $b = 6{\cdot}5$ cm. Illustrate by figures the cases which arise in constructing the triangle, when (i) $a = 7$ cm. (ii) $a = 6$ cm. (iii) $a = 5$ cm. (iv) $a = 4$ cm.

9. Two straight roads, which cross at right angles at A, are carried over a straight canal by bridges at B and C. The distance between the bridges is 461 yards, and the distance from the crossing A to the bridge B is 261 yards. Draw a plan, and by measurement of it ascertain the distance from A to C.

(*Problems. State your construction, and give a theoretical proof*)

10. Draw an isosceles triangle on a base of 4 cm., and having an altitude of 6·2 cm. *Prove* the two sides equal, and measure them to the nearest millimetre.

11. Draw an isosceles triangle having its vertical angle equal to a given angle, and the perpendicular from the vertex on the base equal to a given straight line.

Hence draw an equilateral triangle in which the perpendicular from one vertex on the opposite side is 6 cm. Measure the length of a side to the nearest millimetre.

12. Construct a triangle ABC in which the perpendicular from A on BC is 5·0 cm., and the sides AB, AC are 5·8 cm. and 9·0 cm. respectively. Measure BC.

13. Construct a triangle ABC having the angles at B and C equal to two given angles L and M, and the perpendicular from A on BC equal to a given line P.

14. Construct a triangle ABC (without protractor) having given two angles B and C and the side b.

15. On a given base construct an isosceles triangle having its vertical angle equal to a given angle L.

16. Construct a right-angled triangle, having given the length of the hypotenuse c, and the sum of the remaining sides a and b.

If $c = 5{\cdot}3$ cm., and $a + b = 7{\cdot}3$ cm., find a and b graphically; and calculate the value of $\sqrt{a^2 + b^2}$.

17. Construct a triangle, given the perimeter and the angles at the base. For example, $a + b + c = 12$ cm., $B = 70°$, $C = 80°$.

18. Construct a triangle ABC from the following data:

$a = 6{\cdot}5$ cm., $b + c = 10$ cm., and $B = 60°$.

Measure the lengths of b and c.

19. Construct a triangle ABC from the following data:

$a = 7$ cm., $c - b = 1$ cm., and $B = 55°$.

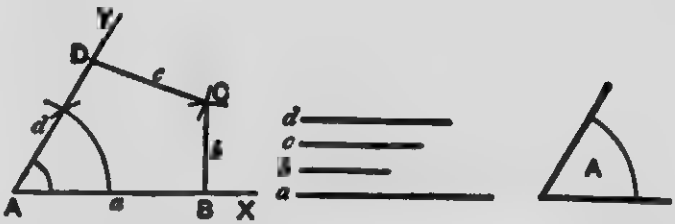
Measure the lengths of b and c.

THE CONSTRUCTION OF QUADRILATERALS

It has been shewn that the shape and size of a triangle are completely determined when the lengths of its three sides are given. A quadrilateral, however, is not completely determined by the lengths of its four sides. From what follows it will appear that *five* independent data are required to construct a quadrilateral.

Problem 11

To construct a quadrilateral, given the lengths of the four sides, and one angle.



Let a, b, c, d be the given lengths of the sides, and A the angle between the sides equal to a and d.

Construction. Take any straight line AX, and cut off from it AB equal to a.

Make the $\angle\ BAY$ equal to the $\angle\ A$.
From AY cut off AD equal to d.

With centre D, and radius c, draw an arc of a circle.

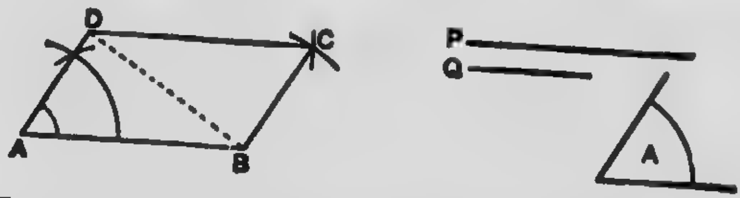
With centre B, and radius b, draw another arc to cut the former at C.

Join DC, BC.

Then $ABCD$ is the required quadrilateral; for by construction the sides are equal to a, b, c, d, and the $\angle\ DAB$ is equal to the given angle.

PROBLEM 12

To construct a parallelogram having given two adjacent sides and the included angle.



Let P and Q be the two given sides, and A the given angle.

Construction 1. (*With ruler and compasses.*) Take a line AB equal to P; and at A make the $\angle\ BAD$ equal to the $\angle\ A$ and make AD equal to Q.

With centre D, and radius P, draw an arc of a circle.

With centre B, and radius Q, draw another arc to cut the former at C.

Then $ABCD$ is the required parᵐ.

Proof.

Join DB.

In the ▲ DCB, BAD,

because $\begin{cases} DC = BA, \\ CB = AD, \\ \text{and } DB \text{ is common;} \end{cases}$

$\therefore$ the $\angle\ CDB$ = the $\angle\ ABD$; *Theor.* 7.

and these are alternate angles,

$\therefore$ DC is parˡ to AB.

Also $DC = AB$;

$\therefore$ DA and BC are also equal and parallel. *Theor.* 20.

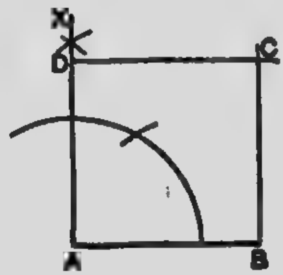
$\therefore$ $ABCD$ is a parᵐ.

Construction 2. (*With set squares.*) Draw AB and AD as before; then with set squares through D draw DC parˡ to AB, and through B draw BC parˡ to AD.

By construction $ABCD$ is a parᵐ having the required parts.

Problem 13

To construct a square on a given side.



Let AB be the given side.

Construction 1. (*With ruler and compasses.*) At A draw AX perp. to AB, and cut off from it AD equal to AB.

With B and D as centres, and with radius AB, draw two arcs cutting at C.

Join BC, DC.

Then $ABCD$ is the required square.

Proof. As in Problem 12, $ABCD$ may be shown to be a par^{m}. And since the $\angle BAD$ is a right angle, the figure is a rectangle. Also, by construction all its sides are equal.

$\therefore$ $ABCD$ is a square.

Construction 2. (*With set squares.*) At A draw AX perp. to AB, and cut off from it AD equal to AB.

Through D draw DC par^{l} to AB, and through B draw BC par^{l} to AD meeting DC in C.

Then, by construction, $ABCD$ is a rectangle. [Def. 3, page 56.]

Also it has the two adjacent sides AB, AD equal.

$\therefore$ it is a square.

EXERCISES

On the Construction of Quadrilaterals

1. Draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure.

Ascertain (without measurement) the number of degrees in each angle, giving a reason for your answer.

2. Draw a square on a side of 2·5 inches. Prove theoretically that its diagonals are equal; and by measuring the diagonals to the nearest hundredth of an inch test the correctness of your drawing.

3. Construct a square on a diagonal of 3·0″, and measure the length of each side. Obtain the average of your results.

4. Draw a parallelogram $ABCD$, having given that one side $AB = 5{\cdot}5$ cm., and the diagonals AC, BD are 8 cm., and 6 cm., respectively. Measure AD.

5. The diagonals of a certain quadrilateral are equal (each 6·0 cm.), and they bisect one another at an angle of 60°. Shew that *five* independent data are here given.

Construct the quadrilateral. Name its species; and give a *formal* proof of your answer. Measure the perimeter. If the angle between the diagonals were increased to 90°, by how much per cent would the perimeter be increased?

6. In a quadrilateral $ABCD$,

$AB = 5{\cdot}6$ cm., $BC = 2{\cdot}5$ cm., $CD = 4{\cdot}0$ cm., and $DA = 3{\cdot}3$ cm.

Shew that the shape of the quadrilateral is not settled by these data.

Draw the quadrilateral when (i) $A = 30°$, (ii) $A = 60°$. Why does the construction fail when $A = 100°$?

Determine graphically the least value of A for which the construction fails.

7. Shew how to construct a quadrilateral, having given the lengths of the four sides and of one diagonal. What conditions must hold among the data in order that the problem may be possible?

Illustrate your method by constructing a quadrilateral $ABCD$, when

(i) $AB = 3{\cdot}0''$, $BC = 1{\cdot}7''$, $CD = 2{\cdot}5''$, $DA = 2{\cdot}8''$, and the diagonal $BD = 2{\cdot}6''$. Measure AC.

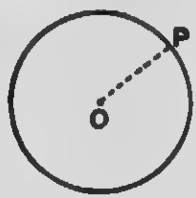
(ii) $AB = 3{\cdot}6$ cm., $BC = 7{\cdot}7$ cm., $CD = 6{\cdot}8$ cm., $DA = 5{\cdot}1$ cm., and the diagonal $AC = 8{\cdot}5$ cm. Measure the angles at B and D.

LOCI

Definition. The **locus** of a point is the path traced out by it when it moves in accordance with some given law.

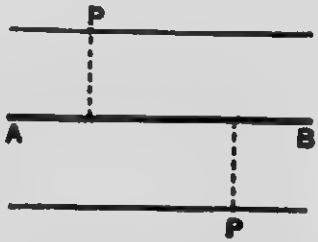
Example 1. Suppose the point P to move so that its distance from a fixed point O is **constant** (say 1 centimetre).

Then the locus of P is evidently the circumference of a circle whose centre is O and radius 1 cm.



Example 2. Suppose the point P moves at a constant distance (say 1 cm.) from a fixed straight line AB.

Then the locus of P is one or other of two straight lines parallel to AB, on either side, and at a distance of 1 cm. from it.

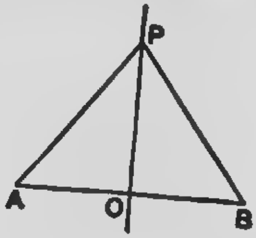


Thus the locus of a point, moving under some given condition, consists of the line or lines to which the point is thereby restricted; provided that the condition is satisfied by every point on such line or lines, and by no other.

When we find a series of points which satisfy the given law, and through which therefore the moving point must pass we are said to **plot the locus** of the point.

PROBLEM 14

To find the locus of a point P which moves so that its distances from two fixed points A and B are always equal to one another.



Here the point P moves through all positions in which $PA = PB$;

$\therefore$ one position of the moving point is at O the middle point of AB.

Suppose P to be *any* other position of the moving point: that is, let $PA = PB$.

Join OP.

Then in the $\triangle$ POA, POB,

because $\begin{cases} PO \text{ is common,} \\ OA = OB, \\ \text{and } PA = PB, \text{ by hypothesis;} \end{cases}$

$\therefore$ the $\angle POA$ = the $\angle POB$. *Theor.* 7.

Hence PO is perpendicular to AB.

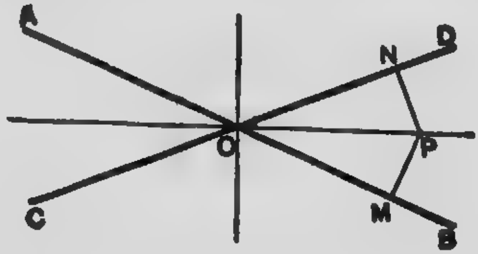
That is, *every point P which is equidistant from A and B lies on the straight line bisecting AB at right angles.*

Likewise it may be proved that every point on the perpendicular through O is equidistant from A and B.

This line is therefore the required locus.

PROBLEM 15

To find the locus of a point P which moves so that its perpendicular distances from two given straight lines AB, CD are equal to one another.



Let P be any point such that the perp. PM = the perp. PN.

Join P to O, the intersection of AB, CD.

Then in the $\triangle$ PMO, PNO,

because $\begin{cases} \text{the } \angle\text{s } PMO, PNO \text{ are } \textit{right angles}, \\ \text{the hypotenuse } OP \text{ is common,} \\ \text{and one side } PM = \text{one side } PN\text{;} \end{cases}$

$\therefore$ the triangles are equal in all respects ; *Theor.* 18.

so that the $\angle\ POM$ = the $\angle\ PON$.

Hence, if P lies within the $\angle\ BOD$, it must be on the bisector of that angle ;

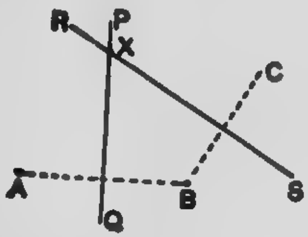
and, if P is within the $\angle\ AOD$, it must be on the bisector of that angle.

It follows that *the required locus is the pair of lines which bisect the angles between AB and CD.*

INTERSECTION OF LOCI

The method of Loci may be used to find the position of a point which is subject to two conditions. For corresponding to each condition there will be a locus on which the required point must lie. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy *both* the given conditions.

EXAMPLE 1. *To find a point equidistant from three given points A, B, C which are not in the same straight line.*

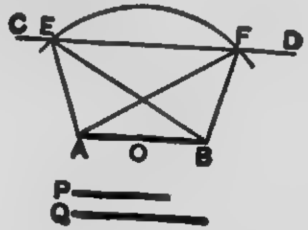


(i) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles.

(ii) Similarly, the locus of points equidistant from B and C is the straight line RS, which bisects BC at right angles.

Hence the point common to PQ and RS must satisfy both conditions: that is to say, X the point of intersection of PQ and RS will be equidistant from A, B, and C.

EXAMPLE 2. *To construct a triangle, having given the base, the altitude, and the length of the median which bisects the base.*



Let AB be the given base, and P and Q the lengths of the altitude and median respectively.

Then the triangle is known if its *vertex* is known.

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P: *then the required vertex must lie on CD.*

(ii) Again, from O the middle point of AB as centre, with radius equal to Q, describe a circle:

then the required vertex must lie on this circle.

Hence any points which are common to CD and the circle satisfy both the given conditions: that is to say, if CD intersect the circle in E, F, each of the points of intersection might be the vertex of the required triangle. This supposes the length of the median Q to be greater than the altitude.

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible.

Obs. In examples on the Intersection of Loci the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible ; and he must observe that if under certain relations *two* solutions are possible, and under other relations no solution exists, there will always be some *intermediate* relation under which the two solutions combine in a single solution.

EXAMPLES ON LOCI

1. Find the locus of a point which moves so that its distance (measured radially) from the circumference of a given circle is constant.

2. A point P moves along a straight line RQ; find the position in which it is equidistant from two given points A and B.

3. A and B are two fixed points within a circle: find points on the circumference equidistant from A and B. How many such points are there?

4. A point P moves along a straight line RQ; find the position in which it is equidistant from two given straight lines AB and CD.

5. A and B are two fixed points 6 cm. apart. Find by the method of loci two points which are 4 cm. distant from A, and 5 cm. from B.

6. AB and CD are two given straight lines. Find points 3 cm. distant from AB, and 4 cm. from CD. How many solutions are there?

7. A straight rod of given length slides between two straight rulers placed at right angles to one another.

Plot the locus of its middle point; and shew that this locus is the fourth part of the circumference of a circle. [See Problem 10.]

8. On a given base as hypotenuse right-angled triangles are described. Find the locus of their vertices.

9. A is a fixed point, and the point X moves on a fixed straight line BC.

Plot the locus of P, the middle point of AX; and *prove* the locus to be a straight line parallel to BC.

10. A is a fixed point, and the point X moves on the circumference of a given circle.

Plot the locus of P, the middle point of AX; and *prove* that this locus is a circle. [See Ex. 3, p. 64.]

11. AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B. If BX revolve about B, find the locus of the middle point of AX.

12. Two straight lines OX, OY cut at right angles, and from P, a point within the angle XOY, perpendiculars PM, PN are drawn to OX, OY respectively. Plot the locus of P when

(i) $PM + PN$ is constant (= 6 cm., say);
(ii) $PM - PN$ is constant (= 3 cm., say).

And in each case give a theoretical proof of the result you arrive at experimentally.

13. Two straight lines OX, OY intersect at right angles at O; and from a movable point P perpendiculars PM, PN are drawn to OX, OY.

Plot (without proof) the locus of P, when

(i) $PM = 2\,PN$;
(ii) $PM = 3\,PN$.

14. Find a point which is at a given distance from a given point and is equidistant from two given parallel straight lines.

When does this problem admit of two solutions, when of one only, and when of none?

15. S is a fixed point 2 inches distant from a given straight line MX. Find two points which are $2\frac{1}{2}$ inches distant from S, and also $2\frac{1}{2}$ inches distant from MX.

16. Find a series of points equidistant from a given point S and a given straight line MX. Draw a curve freehand passing through all the points so found.

17. On a given base construct a triangle of given altitude, having its vertex on a given straight line.

18. Find a point equidistant from the three sides of a triangle.

19. Two straight lines OX, OY cut at right angles; and Q and R points in OX and OY respectively. Plot the locus of the middle point of QR, when

(i) $OQ + OR =$ constant;
(ii) $OQ - OR =$ constant.

20. S and S' are two fixed points. Find a series of points P such that

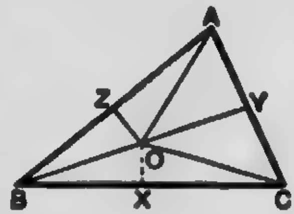
(i) $SP + S'P =$ constant (say 3·5 inches);
(ii) $SP - S'P =$ constant (say 1·5 inches).

In each case draw a curve freehand passing through all the points so found.

ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE

I. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*

Let ABC be a $\triangle$, and X, Y, Z the middle points of its sides.



From Z and Y draw perps. to AB, AC, meeting at O. *Join* OX.

It is required to prove that OX is perp. to BC.

Join OA, OB, OC.

Proof. Because YO bisects AC at right angles,
$\therefore$ it is the locus of points equidistant from A and C;

$$\therefore OA = OC.$$

Again, because ZO bisects AB at right angles,
$\therefore$ it is the locus of points equidistant from A and B;

$$\therefore OA = OB.$$

Hence $OB = OC$.

$\therefore O$ is on the locus of points equidistant from B and C;
that is, OX is perp. to BC.

Hence the perpendiculars from the mid-points of the sides meet at O.

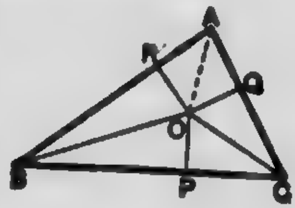
Q.E.D.

II. *The bisectors of the angles of a triangle are concurrent.*

Let ABC be a $\triangle$. Bisect the $\angle$ ABC, BCA by straight lines which meet at O.

Join AO.

It is required to prove that AO bisects the $\angle BAC$.



From O draw OP, OQ, OR perp. to the sides of the $\triangle$.

Proof. Because BO bisects the $\angle ABC$,

$\therefore$ it is the locus of points equidistant from BA and BC;

$$\therefore OP = OR.$$

Similarly CO is the locus of points equidistant from BC and CA;

$$\therefore OP = OQ.$$

Hence $OR = OQ$.

$\therefore$ O is on the locus of points equidistant from AB and AC; that is, OA is the bisector of the $\angle BAC$.

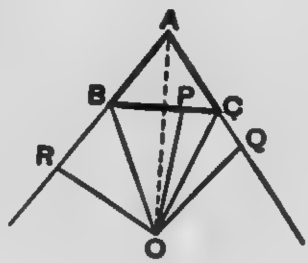
Hence the bisectors of the angles meet at O. Q.E.D.

II*a* *The bisectors of an interior angle at one vertex of a triangle and of the exterior angles at the other vertices are concurrent.*

Let ABC be a $\triangle$, and let AB be produced to D and AC be produced to E. Bisect the $\angle$ CBD, BCE by straight lines which meet at O.

Join AO.

It is required to prove that AO bisects the $\angle BAC$.



From O draw OP, OQ, OR perp. to BC, AE, AD.

Proof. As in Exercise II prove that

$$OP = OR,$$
$$OP = OQ,$$
$$OR = OQ;$$

and hence that the bisectors of the angles BAC, CBD, BCE, meet at O.

Q.E.D.

III. *The medians of a triangle are concurrent.*

Let ABC be a $\triangle$.

Let BY and CZ be two of its medians, and let them intersect at O.

Join AO,

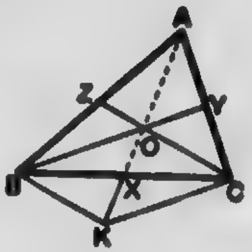
and produce it to meet BC in X.

It is required to shew that AX *is the remaining median of the* $\triangle$.

Through C draw CK parallel to BY;

produce AX to meet CK at K.

Join BK.



Proof. In the $\triangle\ AKC$,

because Y is the middle point of AC, and YO is parallel to CK,

$\therefore$ O is the middle point of AK. *Theor.* 22.

Again in the $\triangle\ ABK$,

since Z and O are the middle points of AB, AK,

$\therefore$ ZO is parallel to BK,

that is, OC is parallel to BK,

$\therefore$ the figure $BKCO$ is a par^m.

But the diagonals of a par^m bisect one another;

$\therefore$ X is the middle point of BC.

That is, AX is a median of the $\triangle$.

Hence the three medians meet at the point O. Q.E.D.

Definition. The point of intersection of the medians is called the **centroid** of the triangle.

Corollary. *The three medians of a triangle cut one another at a point of trisection, the greater segment in each being towards the angular point.*

For in the above figure it has been proved that

$$AO = OK,$$

also that OX is half of OK;

$\therefore$ OX is half of OA:

that is, OX is one third of AX.

Similarly OY is one third of BY,

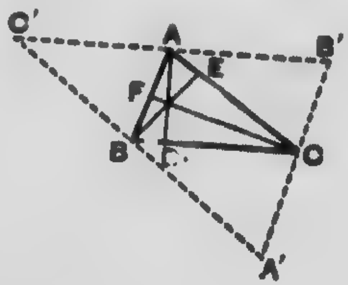
and OZ is one third of CZ.

Q.E.D.

* **IV.** *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

Let ABC be a $\triangle$.

From A, B, and C draw AD, BE, and CF perp. respectively to BC, CA, and AB.



It is required to prove that AD, BE, and CF are concurrent.

Through A draw $C'B'$ parallel to BC.

Through B and C draw $C'A'$ and $A'B'$ parallel respectively to CA and AB.

Proof. Because AC' is parallel to BC,
and BC' is parallel to AC,
$\therefore ACBC'$ is a parallelogram.
$\therefore AC' = BC$.

Similarly we may prove that $AB' = BC$.
$\therefore A$ is the middle point of $C'B'$.
Because the $\angle ADC$ = a right $\angle$,
and the line $B'C'$ is parallel to BC,
$\therefore AD$ is perpendicular to $B'C'$. *Theor.* 14, (1).

Hence AD is perpendicular to $B'C'$ at its middle point.

Similarly, BE and CF are perpendicular to $C'A'$ and $A'B'$ at their middle points.

$\therefore AD$, BE, and CF are concurrent. *Page* 96, I.

Q.E.D.

MISCELLANEOUS PROBLEMS

(*A theoretical proof is to be given in each case.*)

1. A is a given point, and BC a given straight line. From A draw a straight line to make with BC an angle equal to a given angle.
How many such lines can be drawn?

2. Draw the bisector of an angle AOB, without using the vertex O in your construction.

3. P is a given point within the angle AOB. Draw through P a straight line terminated by OA and OB, and bisected at P.

4. OA, OB, OC are three straight lines meeting at O. Draw a transversal terminated by OA and OC, and bisected by OB.

5. Through a given point A draw a straight line so that the part intercepted between two given parallels may be of given length.

When does this problem admit of two solutions? When of only one? And when of none?

6. In a triangle ABC inscribe a rhombus having one of its angles coinciding with the angle A.

7. Use the properties of an equilateral triangle to trisect a given straight line.

8. In any triangle the shorter median bisects the greater side.

(*Construction of Triangles*)

9. Construct a triangle, having given

(i) The middle points of the three sides.

(ii) The lengths of two sides and of the median which bisects the third side.

(iii) The lengths of one side and the medians which bisect the other two sides.

(iv) The lengths of the three medians.

PART II

ON AREAS

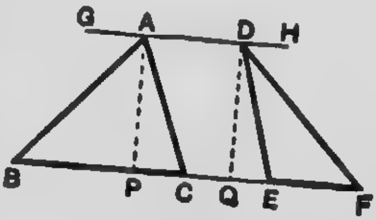
Definitions

1. The **altitude** (or **height**) of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

2. The **altitude** (or **height**) of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base.

NOTE. It is clear that *parallelograms or triangles which are between the same parallels have the same altitude.*

For let AP and DQ be the altitudes of the ▲ ABC, DEF, which are between the same parallels BF, GH.



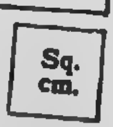
Then the fig. $APQD$ is evidently a rectangle;

$$\therefore AP = DQ.$$

3. The **area** of a figure is the amount of surface contained within its bounding lines.

4. A **square inch** is the area of a square drawn on a side one inch in length.

5. Similarly a **square centimetre** is the area of a square drawn on a side one centimetre in length.

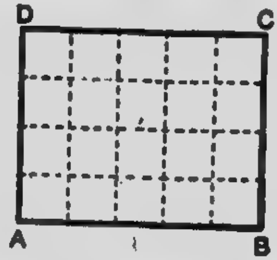


The terms *square yard, square foot, square metre* are to be understood in the same sense.

6. Thus the **unit of area** is the area of a square on a side of unit length.

THEOREM 23

Area of a rectangle. *If the number of units in the length of a rectangle is multiplied by the number of units in its breadth the product gives the number of square units in the area.*



Let $ABCD$ represent a rectangle whose length AB is 5 feet, and whose breadth AD is 4 feet.

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division draw parallels to the sides.

The rectangle $ABCD$ is now divided into compartments, each of which represents one square foot.

Now there are 4 rows, each containing 5 squares,

$\therefore$ the rectangle contains 5×4 square feet.

Similarly, if the length $= a$ linear units, and the breadth $= b$ linear units,

the rectangle contains **ab** *units of* **area.**

And if each side of a square $= a$ linear units,

the square contains $\mathbf{a^2}$ *units of* **area.**

These statements may be thus abridged :

the area of a rectangle = length × breadth.......(i),

the area of a square = (side)^2................(ii).

Q.E.D.

COROLLARIES. (i) *Rectangles which have equal lengths and equal breadths have equal areas.*

(ii) *Rectangles which have equal areas and equal lengths have also equal breadths.*

NOTATION

The rectangle *ABCD* is said to be **contained** by *AB*, *AD*; for these adjacent sides fix its size and shape.

A rectangle whose adjacent sides are *AB*, *AD* is denoted by *rect. AB, AD*, or by *AB, AD*.

A square drawn on the side *AB* is denoted by *sq. on AB*, or by AB^2.

EXERCISES

(On Tables of Length and Area)

1. Draw a figure to shew *why*

(i) 1 sq. yard = 3^2 sq. feet.

(ii) 1 sq. foot = 12^2 sq. inches.

(iii) 1 sq. cm. = 10^2 sq. mm.

2. Draw a figure to shew that the square on a straight line is four times the square on half the line.

3. Use squared paper to shew that the square on $1'' = 10^2$ times the square on $0 \cdot 1''$.

4. If $1''$ represents 5 miles, what does an area of 6 square inches represent?

EXTENSION OF THEOREM 23

The proof of Theorem 23 here given supposes that the length and breadth of the given rectangle are expressed by *whole numbers*; but the formula holds good when the length and breadth are fractional.

This may be illustrated thus:

Suppose the length and breadth are 3·2 cm. and 2·4 cm.; we shall shew that the area is (3·2 × 2·4) sq. cm.

For

$$\text{length} = 3{\cdot}2 \text{ cm.} = 32 \text{ mm.}$$

$$\text{breadth} = 2{\cdot}4 \text{ cm.} = 24 \text{ mm.}$$

$$\therefore \text{area} = (32 \times 24) \text{ sq. mm.} = \frac{32 \times 24}{10^2} \text{ sq. cm.}$$

$$= (3{\cdot}2 \times 2{\cdot}4) \text{ sq. cm.}$$

EXERCISES

(On the Area of a Rectangle)

Draw on squared paper the rectangles of which the length (a) and breadth (b) are given below. Calculate the areas, and verify by the actual counting of squares.

1. $a = 2''$, $b = 3''$.
2. $a = 1{\cdot}5''$, $b = 4''$.
3. $a = 0{\cdot}8''$, $b = 3{\cdot}5''$.
4. $a = 2{\cdot}5''$, $b = 1{\cdot}4''$.
5. $a = 2{\cdot}2''$, $b = 1{\cdot}5''$.
6. $a = 1{\cdot}6''$, $b = 2{\cdot}1''$.

Calculate the areas of the rectangles in which

7. a = 18 metres, b = 11 metres.
8. a = 7 ft., b = 72 in.
9. a = 2·5 km., b = 4 metres.
10. a = ½ mile, b = 1 inch.

11. The area of a rectangle is 30 sq. cm., and its length is 6 cm. Find the breadth. Draw the rectangle on squared paper; and verify your work by counting the squares.

12. Find the length of a rectangle whose area is 3·9 sq. in., and breadth 1·5″. Draw the rectangle on squared paper; and verify your work by counting the squares.

13. (i) When you treble the length of a rectangle without altering its breadth, how many times do you multiply the area?

(ii) When you treble both length and breadth, how many times do you multiply the area?

Draw a figure to illustrate your answers; and state a general rule.

14. In a plan of a rectangular garden the length and breadth are 3·6″ and 2·5″, one inch standing for 10 yards. Find the area of the garden.

If the area is increased by 300 sq. yds., the breadth remaining the same, what will the new length be? And how many inches will represent it on your plan?

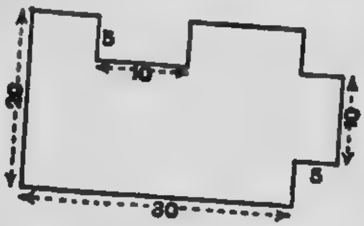
15. Find the area of a rectangular enclosure of which a plan (scale 1 cm. to 20 metres) measures 6·5 cm. by 4·5 cm.

16. The area of a rectangle is 1440 sq. yds. If in a plan the sides of the rectangle are 3·2 cm. and 4·5 cm., on what scale is the plan drawn?

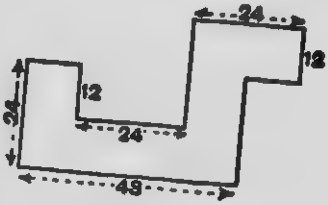
17. The area of a rectangular field is 52,000 sq. ft. On a plan of this, drawn to the scale of 1″ to 100 ft., the length is 3·25″. What is the breadth?

Calculate the areas of the enclosures of which plans are given below. All the angles are right angles, and the dimensions are marked in feet.

18.

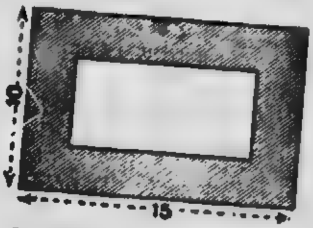


19.



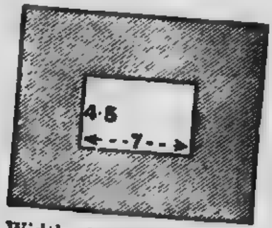
Calculate the areas represented by the shaded parts of the following plans. The dimensions are marked in feet.

20.



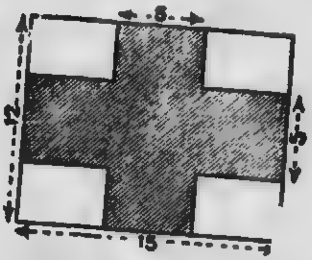
Width of shaded border uniform 2½ ft.

21.

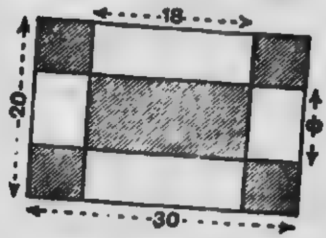


Width of shaded border uniform 4 ft.

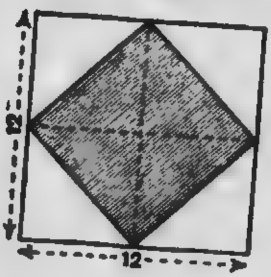
22.



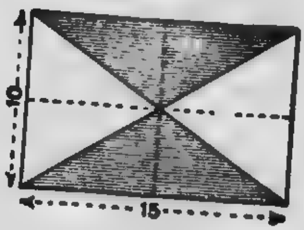
23.



24.

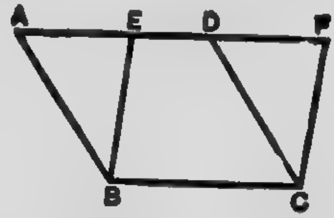


25.



THEOREM 24. [Euclid I. 35]

Parallelograms on the same base and between the same parallels are equal in area.



Let the par$^{\text{ms}}$ $ABCD$, $EBCF$ be on the same base BC, and between the same par$^{\text{ls}}$ BC, AF.

It is required to prove that

the par$^{\text{m}}$ $ABCD$ = the par$^{\text{m}}$ $EBCF$ in area.

Proof. In the ▲ FDC, EAB,

$$\text{because}\left\{\begin{array}{lll} DC = \text{the opp. side } AB\ ; & & \textit{Theor. } 21. \\ \text{the ext. } \angle FDC = \text{the int. opp. } \angle EAB\ ; & & \textit{Theor. } 14. \\ \text{the int. } \angle DFC = \text{the ext. } \angle AEB\ ; & & \\ \therefore \text{ the } \triangle\ FDC = \text{the } \triangle\ EAB. & & \textit{Theor. } 17. \end{array}\right.$$

Now, if from the whole fig. $ABCF$ the △ FDC is taken, the remainder is the par$^{\text{m}}$ $ABCD$.

And if from the whole fig. $ABCF$ the △ EAB is taken, the remainder is the par$^{\text{m}}$ $EBCF$.

∴ these remainders are equal ;

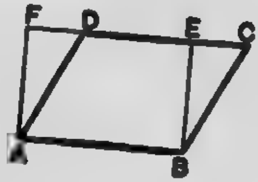
that is, the par$^{\text{m}}$ $ABCD$ = the par$^{\text{m}}$ $EBCF$. Q.E.D.

EXERCISE

In the above diagram the sides AD, EF overlap. Draw diagrams in which (i) these sides do not overlap; (ii) the ends E and D coincide.

Go through the proof with these diagrams, and ascertain if it applies to them without change.

The Area of a Parallelogram



Let $ABCD$ be a parallelogram, and $ABEF$ the rectangle on the same base AB and of the same altitude BE. Then by Theorem 24,

area of par^m $ABCD$ = area of rect. $ABEF$
$= AB \times BE$
= *base* $\times$ *altitude.*

COROLLARY. Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal in area.

EXERCISES

(*Numerical and Graphical*)

1. Find the area of parallelograms in which
   (i) the base = 5·5 cm., and the height = 4 cm.
   (ii) the base = 2·4″, and the height = 1·5″.

2. Draw a parallelogram $ABCD$ having given $AB = 2\frac{1}{2}''$, $AD = 1\frac{1}{2}''$, and the $\angle A = 65°$. Draw and measure the perpendicular from D on AB, and hence calculate the approximate area. Why *approximate?*

Again calculate the area from the length of AD and the perpendicular on it from B. Obtain the average of the two results.

3. Two adjacent sides of a parallelogram are 30 metres and 25 metres, and the included angle is 50°. Draw a plan, 1 cm. representing 5 metres; and by measuring each altitude, make two independent calculations of the area. Give the average result.

4. The area of a parallelogram $ABCD$ is 4·2 sq. in., and the base AB is 2·8″. Find the height. If $AD = 2''$, draw the parallelogram.

5. Each side of a rhombus is 2″, and its area is 3·86 sq. in. Calculate an altitude. Hence draw the rhombus, and measure one of its acute angles.

THEOREM 25

The Area of a Triangle. *The area of a triangle is half the area of the rectangle on the same base and having the same altitude.*

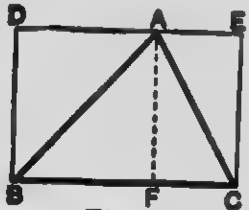


Fig. 1.

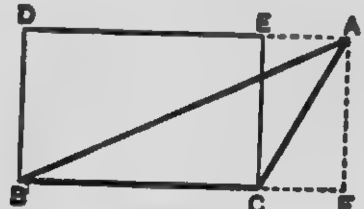


Fig. 2.

Let ABC be a triangle, and $BDEC$ a rectangle on the same base BC and with the same altitude AF.

It is required to prove that the $\triangle$ ABC *is half the rectangle* $BDEC$.

Proof. Since AF is perp. to BC, each of the figures DF, EF is a rectangle.

Because the diagonal AB bisects the rectangle DF,

$\therefore$ the $\triangle$ ABF is half the rectangle DF.

Similarly, the $\triangle$ AFC is half the rectangle FE.

$\therefore$ adding these results in Fig. 1, and taking the difference in Fig. 2,

the $\triangle$ ABC is half the rectangle $BDEC$.

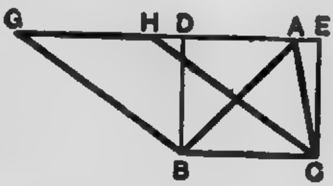
Q.E.D.

COROLLARY. *A triangle is half any parallelogram on the same base and between the same parallels.*

For the $\triangle$ ABC is half the rect. $BCED$.

And the rect. $BCED$ = any par^{m} $BCHG$ on the same base and between the same par^{ls}.

$\therefore$ the $\triangle$ ABC is half the par^{m} $BCHG$.



THE AREA OF A TRIANGLE

If BC and AF respectively contain a units and p units of length, the rectangle $BDEC$ contains ap units of area.

$\therefore$ the area of the $\triangle ABC = \frac{1}{2} ap$ units of area.

This result may be stated thus :

Area of a Triangle $= \frac{1}{2}$ *base* $\times$ *altitude.*

EXERCISES ON THE AREA OF A TRIANGLE

(Numerical and Graphical)

1. Calculate the areas of the triangles in which
   (i) the base = 24 ft., the height = 15 ft.
   (ii) the base = 4·8″, the height = 3·5″.
   (iii) the base = 160 metres, the height = 125 metres.

2. Draw triangles from the following data. In each case draw and measure the altitude with reference to a given side as base: hence calculate the approximate area.
   (i) $a = 8{\cdot}4$ cm., $b = 6{\cdot}8$ cm., $c = 4{\cdot}0$ cm.
   (ii) $b = 5{\cdot}0$ cm., $c = 6{\cdot}8$ cm., $A = 65°$.
   (iii) $a = 6{\cdot}5$ cm., $B = 52°$, $C = 76°$.

3. ABC is a triangle *right-angled* at C; shew that its area $= \frac{1}{2} BC \times CA$.

   Given $a = 6$ cm., $b = 5$ cm., calculate the area.

   Draw the triangle and measure the hypotenuse c; draw and measure the perpendicular from C on the hypotenuse; hence calculate the approximate area.

   Note the error in your approximate result, and express it as a percentage of the true value.

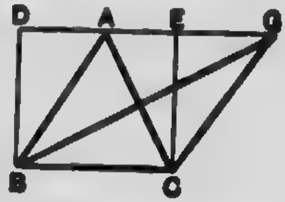
4. Repeat the whole process of the last question for a right-angled triangle ABC, in which $a = 2{\cdot}8''$ and $b = 4{\cdot}5''$; C being the right angle as before.

5. In a triangle, given
   (i) Area = 80 sq. in., base = 1 ft. 8 in.; calculate the altitude.
   (ii) Area = 10·4 sq. cm., altitude = 1·6 cm.; calculate the base.

6. Construct a triangle ABC, having given $a = 3{\cdot}0''$, $b = 2{\cdot}8''$, $c = 2{\cdot}6''$. Draw and measure the perpendicular from A on BC; hence calculate the approximate area.

THEOREM 26. [Euclid I. 37]

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area.



Let the ▲ ABC, GBC be on the same base BC and between the same par[ls] BC, AG.

It is required to prove that

the △ ABC = the △ GBC in area.

Proof. If $BCED$ is the rectangle on the base BC, and between the same parallels as the given triangles,

the △ ABC is half the rect. $BCED$; *Theor.* 25.

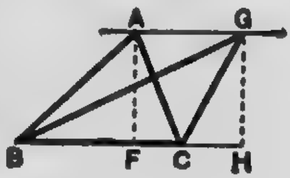
also the △ GBC is half the rect. $BCED$;

∴ the △ ABC = the △ GBC. Q.E.D.

Similarly, *triangles on equal bases and of equal altitudes are equal in area.*

THEOREM 27. [Euclid I. 39]

If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels.



Let the ▲ ABC, GBC, standing on the same base BC, be equal in area ; and let AF and GH be their altitudes.

It is required to prove that AG and BC are par[l].

Proof. The △ ABC is half the rectangle contained by BC and AF ; and the △ GBC is half the rectangle contained by BC and GH ;

∴ the rect. BC, AF = the rect. BC, GH ;

∴ $AF = GH$. *Theor.* 23, *Cor.* 2.

Also AF and GH are par[l] ;

hence AG and FH, that is BC, are par[l]. Q.E.D.

EXERCISES ON THE AREA OF A TRIANGLE

(Theoretical)

1. ABC is a triangle and XY is drawn parallel to the base BC, cutting the other sides at X and Y. Join BY and CX; and shew that

(i) the $\triangle XBC$ = the $\triangle YBC$;
(ii) the $\triangle BXY = \triangle CXY$;
(iii) the $\triangle ABY$ = the $\triangle ACX$.

If BY and CX cut at K, shew that

(iv) the $\triangle BKX$ = the $\triangle CKY$.

2. Shew that a median of a triangle divides it into two parts of equal area.

How would you divide a triangle into *three* equal parts by straight lines drawn from its vertex?

3. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.

4. ABC is a triangle whose base BC is bisected at X. If Y is any point in the median AX, shew that

the $\triangle ABY$ = the $\triangle ACY$ in area.

5. If $ABCD$ is a parallelogram, and BP, DQ are the perpendiculars from B and D on the diagonal AC, then $BP = DQ$.

Also if X is any point in AC, or AC produced,

(i) the $\triangle ADX$ = the $\triangle ABX$;
(ii) the $\triangle CDX$ = the $\triangle CBX$.

6. *The straight line joining the middle points of two sides of a triangle is parallel to the third side.* (Use Theorems 26 and 27.)

7. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to each of the parallel sides.

8. $ABCD$ is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram $ABCD$.

9. If $ABCD$ is a parallelogram, and X, Y any points in DC and AD respectively, the triangles AXB, BYC are equal in area.

10. If $ABCD$ is a parallelogram, and P is any point within it, the sum of the triangles PAB, PCD is equal to half the parallelogram.

EXERCISES ON THE AREA OF A TRIANGLE

(Numerical and Graphical)

1. The sides of a triangular field are 370 yds., 200 yds., and 190 yds. Draw a plan (scale 1″ to 100 yards). Draw and measure an altitude; calculate the approximate area of the field in square yards.

2. Two sides of a triangular enclosure are 124 metres and 144 metres respectively, and the included angle is observed to be 45°. Draw a plan (scale 1 cm. to 20 metres). Make any necessary measurement, and calculate the approximate area.

3. If in a triangle ABC, the area $= 6{\cdot}6$ sq. cm., and the base $BC = 5{\cdot}5$ cm., find the altitude. Hence determine the locus of A.
If also, $BA = 2{\cdot}6$ cm., draw the triangle; and measure CA.

4. In a triangle ABC, given *area* $= 3{\cdot}06$ sq. in., and $a = 3{\cdot}0''$. Find the altitude, and the locus of A. Given $C = 68°$, construct the triangle; and measure b.

5. In a triangle ABC, BC, BA have constant lengths 6 cm. and 5 cm.; BC is fixed, and BA revolves about B. Trace the changes in the area of the triangle as the angle B increases from 0° to 180°.
Answer by drawing a series of triangles, increasing B by increments of 30°. Find their areas and tabulate the results.

(Theoretical)

6. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, shew that the triangles are equal *in area*. Can such triangles ever be *identically* equal?

7. Shew how to draw on the base of a given triangle an isosceles triangle of equal area.

8. If the middle points of the sides of a quadrilateral are joined in order, prove that the *parallelogram* so formed [see Ex. 7, p. 64] is half the quadrilateral.

9. ABC is a triangle, and R, Q the middle points of the sides AB, AC; shew that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral $AQXR$.

10. Two triangles of equal area stand on the same base but on opposite sides of it: shew that the straight line joining their vertices is bisected by the base, or by the base produced.

[The method given below may be omitted from a first course. In any case it must be postponed till Theorem 29 has been read.]

The Area of a Triangle. *Given the three sides of a triangle, to calculate the area.*

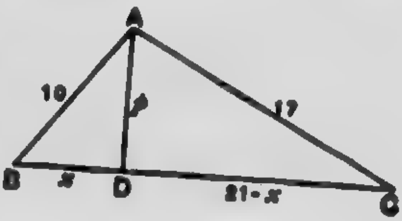
Example. *Find the area of a triangle whose sides measure* 21 m., 17 m., *and* 10 m.

Let ABC represent the given triangle.

Draw AD perp. to BC, and denote AD by p.

We shall first find the length of BD.

Let $BD = x$ metres; then $DC = 21 - x$ metres.



From the right-angled $\triangle ADB$, we have by Theorem 29

$$AD^2 = AB^2 - BD^2 = 10^2 - x^2.$$

And from the right-angled $\triangle ADC$,

$$AD^2 = AC^2 - DC^2 = 17^2 - (21 - x)^2;$$

$$\therefore 10^2 - x^2 = 17^2 - (21 - x)^2$$

or, $$100 - x^2 = 289 - 441 + 42x - x^2;$$

whence $$x = 6.$$

Again, $$AD^2 = AB^2 - BD^2;$$

or $$p^2 = 10^2 - 6^2 = 64;$$

$$\therefore p = 8.$$

Now *Area of triangle* $= \frac{1}{2}$ *base* $\times$ *altitude*

$= (\frac{1}{2} \times 21 \times 8)$ sq. m. $= 84$ sq. m.

EXERCISES

Find the area of the triangles, whose sides are as follows:

1. 20 ft., 13 ft., 11 ft.
2. 15 yds., 14 yds., 13 yds.
3. 21 m., 20 m., 13 m.
4. 30 cm., 25 cm., 11 cm.
5. 37 ft., 30 ft., 13 ft.
6. 51 m., 37 m., 20 m.
7. If the given sides are a, b, and c units in length, prove

(i) $x = \dfrac{a^2 + c^2 - b^2}{2a}$; (ii) $p^2 = c^2 - \left\{\dfrac{a^2 + c^2 - b^2}{2a}\right\}^2$;

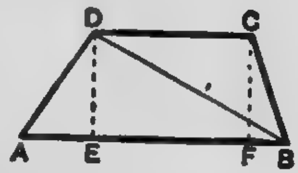
(iii) $\triangle = \frac{1}{4}\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}$

THE AREA OF QUADRILATERALS

Theorem 28

To find the area of (i) *a trapezium.*
(ii) *any quadrilateral.*

(i) Let $ABCD$ be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB.



Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units.

Then the area of $ABCD = \triangle ABD + \triangle DBC$

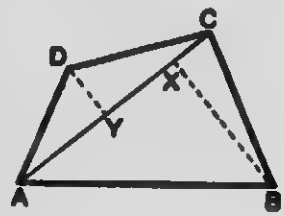
$$= \tfrac{1}{2} AB \cdot DE + \tfrac{1}{2} DC \cdot CF$$

$$= \frac{1}{2} ah + \frac{1}{2} bh = \frac{h}{2}(a+b).$$

That is,

the area of a trapezium = ½ height × (*the sum of the parallel sides*).

(ii) Let $ABCD$ be any quadrilateral. Draw a diagonal AC; and from B and D draw perpendiculars BX, DY to AC. These perpendiculars are called **offsets**.



If AC contains d units of length, and BX, DY p and q units respectively,

the area of the quad¹ $ABCD = \triangle ABC + \triangle ADC$

$$= \tfrac{1}{2} AC \cdot BX + \tfrac{1}{2} AC \cdot DY$$

$$= \tfrac{1}{2} dp + \tfrac{1}{2} dq = \tfrac{1}{2} d(p+q).$$

That is to say,

the area of a quadrilateral = ½ *diagonal* × (*sum of offsets*).

EXERCISES

(Numerical and Graphical)

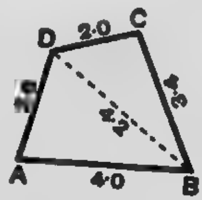
1. Find the area of the trapezium in which the two parallel sides are 4·7″ and 3·3″, and the height 1·5″.

2. In a quadrilateral $ABCD$, the diagonal $AC = 17$ feet; and the offsets from it to B and D are 11 feet and 9 feet. Find the area.

3. In a plan $ABCD$ of a quadrilateral enclosure, the diagonal AC measures 8·2 cm., and the offsets from it to B and D are 3·4 cm. and 2·6 cm. respectively. If 1 cm. in the plan represents 5 metres, find the area of the enclosure.

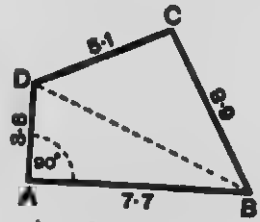
4. Draw a quadrilateral $ABCD$ from the adjoining rough plan, the dimensions being given in inches.

Draw and measure the offsets to A and C from the diagonal BD; and hence calculate the area of the quadrilateral.



5. Draw a quadrilateral $ABCD$ from the details given in the adjoining plan. The dimensions are to be in centimetres.

Make any necessary measurements of your figure, and calculate its area.



6. Draw a trapezium $ABCD$ from the following data: AB and CD are the parallel sides. $AB = 4''$; $AD = BC = 2''$; the $\angle A$ = the $\angle B = 60°$.

Make any necessary measurements, and calculate the area.

7. Draw a trapezium $ABCD$ in which AB and CD are the parallel sides; and $AB = 9$ cm., $CD = 3$ cm., and $AD = BC = 5$ cm.

Make any necessary measurement, and calculate the area.

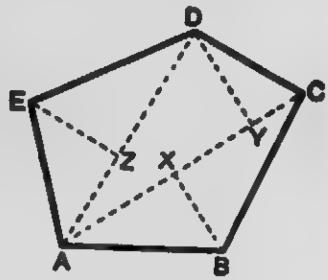
8. From the formula *area of quadl. = ½ diag. × (sum of offsets)* shew that, if the diagonals are *at right angles*,

area = ½ (product of diagonals).

9. Given the lengths of the diagonals of a quadrilateral, and the angle between them, prove that the area is the same wherever they intersect.

THE AREA OF ANY RECTILINEAL FIGURE

1st Method. A rectilineal figure may be divided into triangles whose areas can be separately calculated from suitable measurements. The sum of these areas will be the area of the given figure.

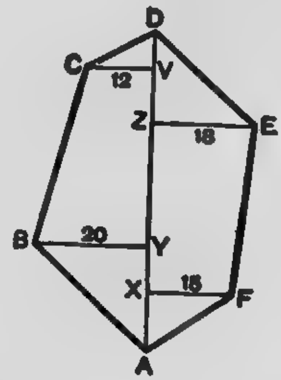


Example. The measurements required to find the area of the figure $ABCDE$ are AC, AD, and the offsets BX, DY, EZ.

2d Method. The area of a rectilineal figure is also found by taking a **base-line** (AD in the diagram below) and offsets from it. These divide the figure into *right-angled* triangles and *right-angled* trapeziums, whose areas may be found after measuring the offsets and the various sections of the base-line.

Example. Find the area of the enclosure $ABCDEF$ from the plan and measurements tabulated below.

	Yards.	
	$AD = 56$	
$VC = 12$	$AV = 50$	
	$AZ = 40$	$ZE = 18$
$YB = 20$	$AY = 18$	
	$AX = 10$	$XF = 15$



The measurements are made from A along the base-line to the points from which the offsets spring.

Here $\triangle AXF = \frac{1}{2} AX \times XF = \frac{1}{2} \times 10 \times 15 =$ 75 sq. yds

$\triangle AYB = \frac{1}{2} AY \times YB = \frac{1}{2} \times 18 \times 20 =$ 180

$\triangle DZE = \frac{1}{2} DZ \times ZE = \frac{1}{2} \times 16 \times 18 =$ 144

$\triangle DVC = \frac{1}{2} DV \times VC = \frac{1}{2} \times 6 \times 12 =$ 36

trap$^m XFEZ = \frac{1}{2} XZ \times (XF + ZE) = \frac{1}{2} \times 30 \times 33 =$ 495

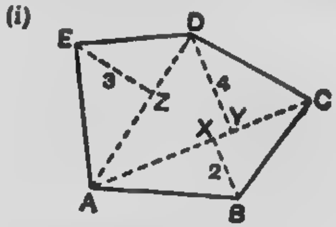
trap$^m YBCV = \frac{1}{2} YV \times (YB + VC) = \frac{1}{2} \times 32 \times 32 =$ 512

$\therefore$, by addition, the fig. $ABCDEF =$ 1442 sq. yds.

EXERCISES

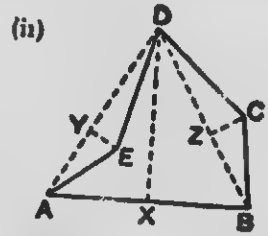
1. Calculate the areas of the figures (i) and (ii) from the plans and dimensions (in cms.) given below.

(i)



AC=6 cm., AD=5 cm.
Lengths of offsets figured in diagram.

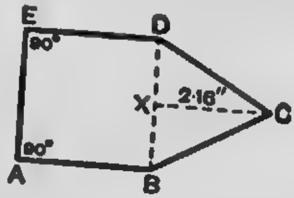
(ii)



AB=BD=DA=6 cm.
EY=CZ=1 cm.
DX=5·2 cm.

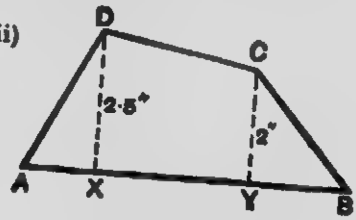
2. Draw full size the figures whose plans and dimensions are given below; and calculate the area in each case.

(i)



The fig. is equilateral: each side to be $2\frac{1}{2}''$.

(ii)

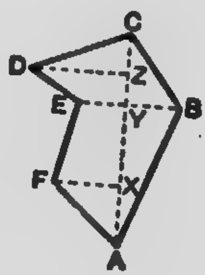


$AX = 1\frac{1}{4}''$, $XY = 2\frac{3}{4}''$, $YB = 1\frac{3}{4}''$

3. Find the area of the figure *ABCDEF* from the following measurements and draw a plan in which 1 cm. represents 20 metres.

	Metres.	
	to C	
	180	
80 to D	150	
40 to E	120	50 to B
60 to F	50	
	From A	

The Plan.



EXERCISES ON QUADRILATERALS

(Theoretical)

1. *ABCD* is a rectangle, and *PQRS* the figure formed by joining in order the middle points of the sides.

Prove (i) that *PQRS* is a rhombus;

(ii) that the area of *PQRS* is half that of *ABCD*.

Hence shew that *the area of a rhombus is half the product of its diagonals.*

Is this true of any quadrilateral whose diagonals cut at right angles? Illustrate your answer by a diagram.

2. Prove that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals.

Hence shew how a parallelogram *ABCD* may be bisected by a straight line drawn

(i) through a given point *P*;
(ii) perpendicular to the side *AB*;
(iii) parallel to a given line *QR*.

3. In the trapezium *ABCD*, *AB* is parallel to *DC*; and *X* is the middle point of *BC*. Through *X* draw *PQ* parallel to *AD* to meet *AB* and *DC* produced at *P* and *Q*. Then prove

(i) trapezium *ABCD* = par^{m} *APQD*.
(ii) trapezium *ABCD* = twice the $\triangle$ *AXD*.

(Graphical)

4. The diagonals of a quadrilateral *ABCD* cut at right angles, and measure 3·0″ and 2·2″ respectively. Find the area.

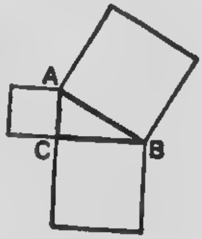
Shew by a figure that the area is the same wherever the diagonals cut, so long as they are at right angles.

5. In the parallelogram *ABCD*, *AB* = 8·0 cm., *AD* = 3·2 cm., and the perpendicular distance between *AB* and *DC* = 3·0 cm. Draw the parallelogram. Calculate the distance between *AD* and *BC*; and check your result by measurement.

6. One side of a parallelogram is 2·5″, and its diagonals are 3·4″ and 2·4″. Construct the parallelogram; and, after making any necessary measurement, calculate the area.

7. *ABCD* is a parallelogram on a fixed base *AB* and of constant area. Find the locus of the intersection of its diagonals.

EXERCISES LEADING TO THEOREM 29



In the adjoining diagram, ABC is a triangle *right-angled* at C; and squares are drawn on the three sides. Let us compare the area of the square on the hypotenuse AB with the *sum* of the squares on the sides AC, CB which contain the right angle.

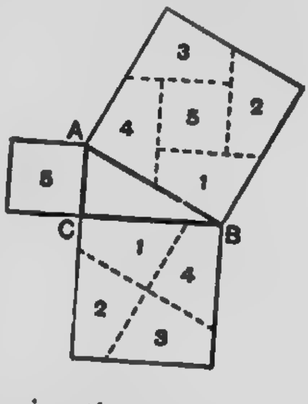
1. Draw the above diagram, making $AC = 3$ cm., $BC = 4$ cm.;

Then the area of the square on $AC = 3^2$, or 9 sq. cm.
and............the square on $BC = 4^2$, or 16 sq. cm.
∴ the sum of the squares on AC, $BC =$ 25 sq. cm.

Now *measure* AB; hence calculate the area of the square on AB, and compare the result with the *sum* already obtained.

2. Repeat the above process, making $AC = 1{\cdot}0''$, $BC = 2{\cdot}4''$.

3. If $a = 15$, $b = 8$, $c = 17$, shew arithmetically that $c^2 = a^2 + b^2$. Now draw on squared paper a triangle ABC, whose sides a, b, and c are 15, 8, and 17 units of length; and *measure the angle* ACB.



4. Take any triangle ABC, right-angled at C; and draw squares on AC, CB, and on the hypotenuse AB.

Through the mid-point of the square on CB (*i.e.* the intersection of the diagonals) draw lines parallel and perpendicular to the hypotenuse, thus dividing the square into four *congruent* quadrilaterals. These, together with the square on AC, will be found exactly to fit into the square on AB, in the way indicated by corresponding numbers.

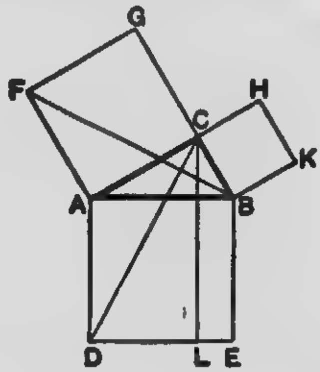
These experiments point to the conclusion that:

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

A formal proof of this theorem is given on the next page.

THEOREM 29. [Euclid I. 47]

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled $\triangle$, having the angle ACB a rt. $\angle$.

It is required to prove that the square on the hypotenuse AB = the sum of the squares on AC, CB.

On AB describe the sq. $ADEB$; and on AC, CB describe the sqq. $ACGF$, $CBKH$.

Through C draw CL par[l] to AD or BE.

Join CD, FB.

Proof. Because each of the $\angle$s ACB, ACG is a rt. $\angle$,

$\therefore$ BC and CG are in the same st. line.

Now the rt. $\angle\ BAD$ = the rt. $\angle\ FAC$;

add to each the $\angle\ CAB$:

then the whole $\angle\ CAD$ = the whole $\angle\ FAB$.

Then in the $\triangle$s CAD, FAB,

$$\text{because}\left\{\begin{array}{l} CA = FA, \\ AD = AB, \\ \text{and the included } \angle CAD = \text{the included } \angle FAB; \end{array}\right.$$

$\therefore$ the $\triangle\ CAD$ = the $\triangle\ FAB$. *Theor.* 4.

Now the rect. AL is double of the $\triangle\, CAD$, being on the same base AD, and between the same par^{ls} AD, CL.

And the sq. GA is double of the $\triangle\, FAB$, being on the same base FA, and between the same par^{ls} FA, GB.

$\therefore$ the rect. AL = the sq. GA.

Similarly by joining CE, AK, it can be shewn that
the rect. BL = the sq. HB.

$\therefore$ the whole sq. AE = the sum of the sqq. GA, HB:

that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB.

Q.E.D.

Obs. This is known as the Theorem of Pythagoras. The result established may be stated as follows:

$$AB^2 = BC^2 + CA^2.$$

That is, if a and b denote the lengths of the sides containing the right angle; and if c denotes the hypotenuse,

$$c^2 = a^2 + b^2.$$

Hence $\qquad a^2 = c^2 - b^2$; and $\quad b^2 = c^2 - a^2$.

NOTE 1. The following important results should be noticed.
If CL and AB intersect in O, it has been shewn in the course of the proof that

the sq. GA = the rect. AL;
that is, AC^2 = the rect. contained by AB, AO. . . . . (i)

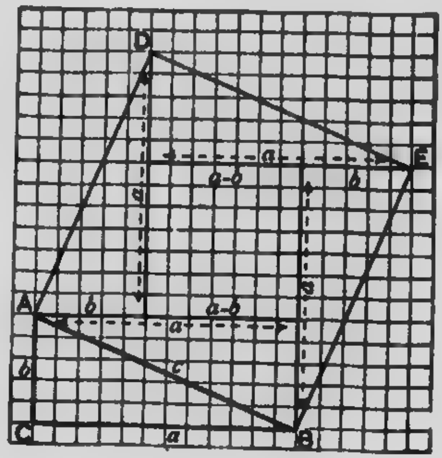
Also the sq. HB = the rect. BL;
that is, BC^2 = the rect. contained by BA, BO. . . . . (ii)

NOTE 2. It can be proved by superposition that *squares standing on equal sides are equal in area.*

Also we can prove conversely,

If two squares are equal in area they stand on equal sides.

EXPERIMENTAL PROOFS OF PYTHAGORAS'S THEOREM

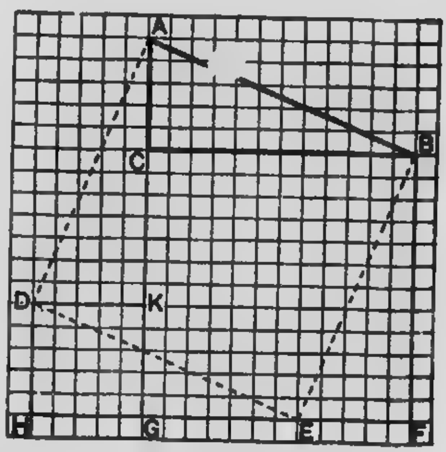


I. Here *ABC* is the given rt.-angled △; and *ABED* is the square on the hypotenuse *AB*.

By drawing lines par[l] to the sides *BC*, *CA*, it is easily seen that the sq. *BD* is divided into 4 rt.-angled ▲, each identically equal to *ABC*, together with a central square.

Hence

sq. on hypotenuse $c = 4$ rt. ∠[d]▲
+ the central square
$= 4 \cdot \frac{1}{2}\, ab + (a - b)^2$
$= 2ab + a^2 - 2ab + b^2$
$= a^2 + b^2.$



II. Here *ABC* is the given rt.-angled △, and the figs. *CF*, *HK* are the sqq. on *CB*, *CA* placed side by side.

FE is made equal to *DH* or *CA*; and the two sqq. *CF*, *HK* are cut along the lines *BE*, *ED*.

Then it will be found that the △ *DHE* may be placed so as to fill up the space *ACB*; and the △ *BFE* may be made to fill the space *AKD*.

Hence the two sqq. *CF*, *HK* may be fitted together so as to form the single fig. *ABED*, which will be found to be a perfect square, namely the square on the hypotenuse *AB*.

EXERCISES

(Numerical and Graphical)

1. Draw a triangle ABC, right-angled at C, having given

(i) $a = 3$ cm., $b = 4$ cm.;
(ii) $a = 2{\cdot}5$ cm., $b = 6{\cdot}0$ cm.;
(iii) $a = 1{\cdot}2''$, $b = 3{\cdot}5''$.

In each case calculate the length of the hypotenuse c, and verify your result by measurement.

2. Draw a triangle ABC, right-angled at C, having given:

(i) $c = 3{\cdot}4''$, $a = 3{\cdot}0''$; [See Problem 10]
(ii) $c = 5{\cdot}3$ cm., $b = 4{\cdot}5$ cm.

In each case calculate the remaining side, and verify your result by measurement.

(The following examples are to be solved by calculation; but in each case a plan should be drawn on some suitable scale, and the calculated result verified by measurement.)

3. A ladder whose foot is 9 feet from the front of a house reaches to a window-sill 40 feet above the ground. What is the length of the ladder?

4. A ship sails 33 miles due South, and then 56 miles due West. How far is it then from its starting point?

5. Two ships are observed from a signal station to bear respectively N.E. 6·0 km. distant, and N.W. 1·1 km. distant. How far are they apart?

6. A ladder 65 feet long reaches to a point in the face of a house 63 feet above the ground. How far is the foot from the house?

7. B is due East of A, but at an unknown distance. C is due South of B, and distant 55 metres. If AC is 73 metres, find AB.

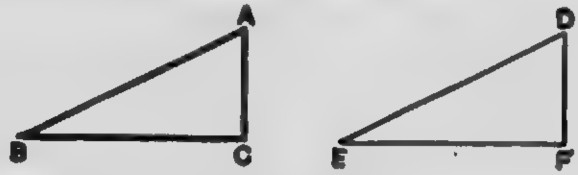
8. A man travels 27 miles due South; then 24 miles due West; finally 20 miles due North. How far is he from his starting point?

9. From A go West 25 metres, then North 60 metres, then East 80 metres, finally South 12 metres. How far are you then from A?

10. A ladder 50 feet long is placed so as to reach a window 48 feet high; and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

THEOREM 30. [Euclid I. 48]

If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle.



Let ABC be a triangle in which

the sq. on AB = the sum of the sqq. on BC, CA.

It is required to prove that ACB *is a right angle.*

Make EF equal to BC.

Draw FD perp. to EF, and make FD equal to CA.

Join ED.

Proof. Because $EF = BC$,

$\therefore$ the sq. on EF = the sq. on BC.

And because $FD = CA$,

$\therefore$ the sq. on FD = the sq. on CA.

Hence the sum of the sqq. on EF, FD = the sum of the sqq. on BC, CA.

But since EFD is a rt. $\angle$

$\therefore$ the sum of the sqq. on EF, FD = the sq. on DE: *Theor.* 29.

And, by hypothesis, the sqq. on BC, CA = the sq. on AB.

$\therefore$ the sq. on DE = the sq. on AB.

$\therefore DE = AB$.

Then in the $\triangle$ ACB, DFE,

because $AC = DF$, $CB = FE$, and $AB = DE$;

$\therefore$ the $\angle ACB$ = the $\angle DFE$. *Theor.* 7.

But, by construction, DFE is a right angle;

$\therefore$ the $\angle ACB$ is a right angle. Q.E.D.

EXERCISES ON THEOREMS 29, 30

(Theoretical)

1. Shew that the square on the diagonal of a given square is double of the given square.

2. In the $\triangle ABC$, AD is drawn perpendicular to the base BC. If the side c is greater than b, shew that $c^2 - b^2 = BD^2 - DC^2$.

3. If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively: shew that

$$AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2.$$

4. ABC is a triangle right-angled at A; and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined. Prove that

$$BQ^2 + PC^2 = BC^2 + PQ^2.$$

5. In a right-angled triangle four times the sum of the squares on the medians drawn from the acute angles is equal to five times the square on the hypotenuse.

6. Describe a square equal to the sum of two given squares.

7. Describe a square equal to the difference between two given squares.

8. Divide a straight line into two parts so that the square on one part may be twice the square on the other.

9. Divide a straight line into two parts such that the sum of their squares shall be equal to a given square.

(Numerical and Graphical)

10. Determine which of the following triangles are right-angled:

(i) $a = 14$ cm., $b = 48$ cm., $c = 50$ cm.;
(ii) $a = 40$ cm., $b = 10$ cm., $c = 41$ cm.;
(iii) $a = 20$ cm., $b = 99$ cm., $c = 101$ cm.

11. ABC is an isosceles triangle right-angled at C; deduce from Theorem 29 that $AB^2 = 2AC^2$.

Illustrate this result graphically by drawing both diagonals of the square on AB, and one diagonal of the square on AC.

If $AC = BC = 2''$, find AB to the nearest hundredth of an inch, and verify your calculation by actual construction and measurement.

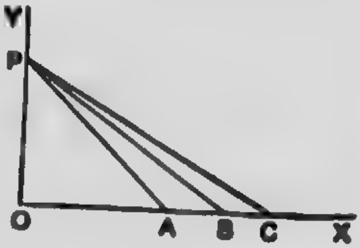
12. Draw a square on a diagonal of 6 cm. Calculate, and also measure, the length of a side. Find the area.

Problem 16

To draw squares whose areas shall be respectively twice, three times, four times, ..., that of a given square.

Hence find graphically approximate values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, ...

Take OX, OY at right angles to one another, and from them mark off OA, OP, each *one* unit of length. Join PA.



$$\text{Then } PA^2 = OP^2 + OA^2 = 1 + 1 = 2.$$
$$\therefore PA = \sqrt{2}.$$

From OX mark off OB equal to PA, and join PB;

$$\text{then } PB^2 = OP^2 + OB^2 = 1 + 2 = 3.$$
$$\therefore PB = \sqrt{3}.$$

From OX mark off OC equal to PB, and join PC;

$$\text{then } PC^2 = OP^2 + OC^2 = 1 + 3 = 4.$$
$$\therefore PC = \sqrt{4}.$$

The lengths of PA, PB, PC may now be found *by measurement*; and by continuing the process we may find $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, ...

EXERCISES ON THEOREMS 29, 30 (*Continued*)

13. Prove the following formula:

Diagonal of square = side $\times \sqrt{2}$.

Hence find to the nearest centimetre the diagonal of a square on a side of 50 metres.

Draw a plan (scale 1 cm. to 10 metres) and [illegible]ain the result as nearly as you can by measurement.

14. ABC is an equilateral triangle of which each side $= 2m$ units, and the perpendicular from any vertex to the opposite side $= p$.

Prove that $p = m\sqrt{3}$.

Test this result graphically, when each side $= 8$ cm.

15. If in a triangle $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$; prove algebraically that $c^2 = a^2 + b^2$.

Hence by giving various numerical value to m and n, find sets of numbers representing the sides of right-angled triangles.

16. In a triangle ABC, AD is drawn perpendicular to BC. Let p denote the length of AD.

(i) If $a = 25$ cm., $p = 12$ cm., $BD = 9$ cm.; find b and c.

(ii) If $b = 41''$, $c = 50''$, $BD = 30''$; find p and a.

And prove that $\sqrt{b^2 - p^2} + \sqrt{c^2 - p^2} = a$.

17. In the triangle ABC, AD is drawn perpendicular to BC. Prove that

$$c^2 - BD^2 = b^2 - CD^2.$$

If $a = 51$ cm., $b = 20$ cm., $c = 37$ cm.; find BD.

Thence find p, the length of AD, and the area of the triangle ABC.

18. Find by the method of the last example the areas of the triangles whose sides are as follows:

(i) $a = 17''$, $b = 10''$, $c = 9''$.

(ii) $a = 25$ ft., $b = 17$ ft., $c = 12$ ft.

(iii) $a = 41$ cm., $b = 28$ cm., $c = 15$ cm.

(iv) $a = 40$ yd., $b = 37$ yd., $c = 13$ yd.

19. A straight rod PQ slides between two straight rulers OX, OY placed at right angles to one another. In one position of the rod $OP = 5{\cdot}6$ cm., and $OQ = 3{\cdot}3$ cm. If in another position $OP = 4{\cdot}0$ cm., find OQ graphically; and test the accuracy of your drawing by calculation.

20. ABC is a triangle right-angled at C, and p is the length of the perpendicular from C on AB. By expressing the area of the triangle in two ways, shew that

$$pc = ab.$$

Hence deduce $$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

1. A rectangle $ABCD$ is said to be **contained** by two adjacent sides AB, AD; for these sides fix its size and shape.

A rectangle whose adjacent sides are AB, AD is denoted by *the rect.* AB, AD; or by AB, AD.

Similarly a square drawn on the side AB is denoted by *the sq. on* AB, or AB^2.

Geometrical illustration of algebraic identities.

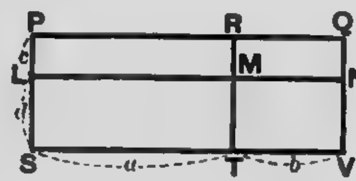
A. *Geometrical illustration of* $(a + b)\, k = ak + lk$.

Let $ST = a$ units of length,
$TV = b$ units of length,
and $PS = k$ units of length.

$$\begin{aligned} \text{Then Area of } SP, PQ &= k(a + b) && Th.\ 23. \\ \text{Area of } SP, PR &= ka && Th.\ 23. \\ \text{Area of } TR, RQ &= kb && Th.\ 23. \\ \therefore\ k(a + b) &= ak + bk. \end{aligned}$$

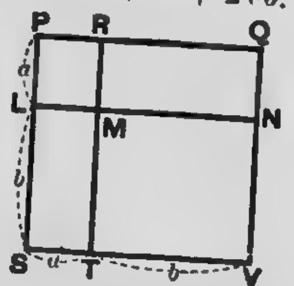
B. *Geometrical illustration of* $(a + b)(c + d) = ac + ad + bc + bd$.

Let $ST = a$ units of length.
Let $TV = b$ units of length.
Let $PL = c$ units of length.
And $LS = d$ units of length.



$$\begin{aligned} \text{Then Area of } SP, PQ &= (c + d)(a + b). && Th.\ 23. \\ \text{Area of } LP, PR &= ac && Th.\ 23. \\ \text{Area of } MR, RQ &= cb && Th.\ 23. \\ \text{Area of } SL, LM &= da && Th.\ 23. \\ \text{Area of } TM, MN &= db && Th.\ 23. \\ \therefore\ (a + b)(c + d) &= ac + bc + ad + bd. \end{aligned}$$

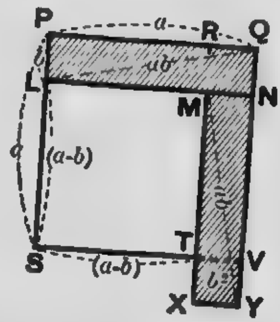
C. *Geometrical illustration of* $(a+b)^2 = a^2 + b^2 + 2ab$.



The area SP, PQ = the area LP, PR + the area TM, MN + the area SL, LM + the area MR, RQ.

Hence $(a+b)^2 = a^2 + b^2 + 2ab$.

D. *Geometrical illustration of* $(a-b)^2 = a^2 + b^2 - 2ab$.



Let $PQ = a$ units of length,

$RQ = b$ units of length,

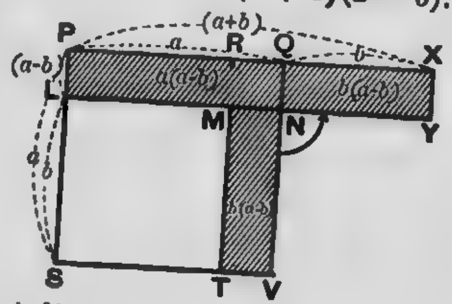
then $PR = (a-b)$ units of lengths.

Hence :

the area SL, LM = the area SP, PQ + the area XT, TV − the area LP, PQ − the area XM, MN.

or $(a-b)^2 = a^2 + b^2 - 2ab$.

E. *Geometrical illustration of* $a^2 - b^2 = (a+b)(a-b)$.



Area $SP, PQ - SL, LM$

= Gnomon P, N, T

= the area LP, PQ + the area TM, MN

= the area LP, PQ + the area NQ, QX

= the area LP, PX.

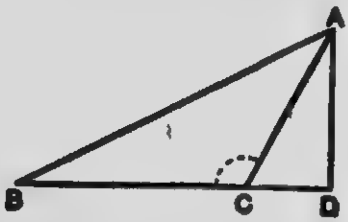
Hence $a^2 - b^2 = (a-b)(a+b)$.

EXERCISES

1. Illustrate $k(a+b+c+d+e) = ak + bk + ck + dk + ek$.
2. Illustrate $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
3. Illustrate A, B and C (above) by paper-folding exercises.

THEOREM 31. [Euclid II. 12]

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle **together with** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*



Let ABC be a triangle obtuse-angled at C; and let AD be drawn perp. to BC produced, so that CD is the projection of the side CA on BC. [See Def. p. 63.]

It is required to prove that

$$AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD.$$

Proof. Because BD is the *sum* of the lines BC, CD,

$\therefore\ BD^2 = BC^2 + CD^2 + 2\,BC \cdot CD.$ *Page* 129, C.

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) + 2\,BC \cdot CD.$

$$\left.\begin{array}{l}\text{But } BD^2 + DA^2 = AB^2 \\ \text{and } CD^2 + DA^2 = CA^2\end{array}\right\}, \text{ for the } \angle D \text{ is a rt. } \angle.$$

Hence $\quad AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD.$

Q.E.D.

Theorem 32. [Euclid II. 13]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle **diminished by** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*

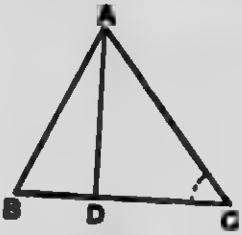


Fig. 1.

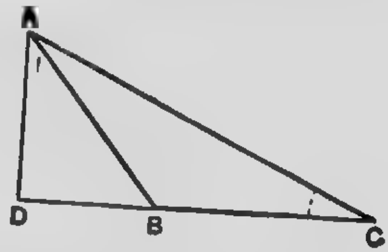


Fig. 2.

Let ABC be a triangle in which the $\angle C$ is acute; and let AD be drawn perp. to BC, or BC produced; so that CD is the projection of the side CA on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD.$$

Proof. Since in both figures BD is the *difference* of the lines BC, CD,

$$\therefore\ BD^2 = BC^2 + CD^2 - 2\,BC \cdot CD. \qquad \textit{Page } 129, \text{D}$$

To each of these equals add DA^2.

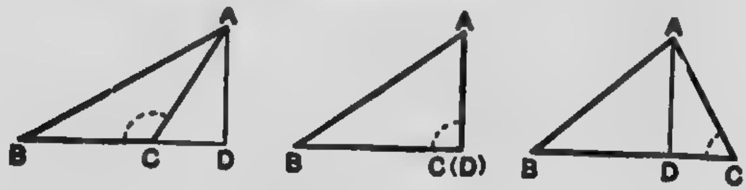
Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) - 2\,BC \cdot CD$.....(i)

$$\left.\begin{aligned} \text{But } BD^2 + DA^2 &= AB^2 \\ \text{and } CD^2 + DA^2 &= CA^2 \end{aligned}\right\}, \text{ for the } \angle D \text{ is a rt. } \angle.$$

Hence $AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD$.

Q.E.D.

SUMMARY OF THEOREMS 29, 31, AND 32.



(i) If the $\angle\ ACB$ is *obtuse*,
$$AB^2 = BC^2 + CA^2 + 2\,BC \cdot CD. \qquad \textit{Theor. } 31.$$
(ii) If the $\angle\ ACB$ is *a right angle*,
$$AB^2 = BC^2 + CA^2. \qquad \textit{Theor. } 29.$$
(iii) If the $\angle\ ACB$ is *acute*,
$$AB^2 = BC^2 + CA^2 - 2\,BC \cdot CD. \qquad \textit{Theor. } 32.$$

Observe that in (i) or (ii), if the $\angle ACB$ becomes 90°, AD coincides with AC, and CD (the projection of CA) vanishes; hence, in this case, $2\,BC \cdot CD = 0$.

Thus the three results may be collected in one enunciation:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle contained by those sides is obtuse, a right angle, or acute; the difference in cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

EXERCISES

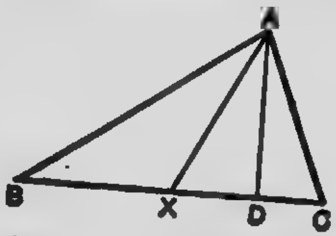
1. In a triangle ABC, $a = 21$ cm., $b = 17$ cm., $c = 10$ cm. By how many square centimetres does c^2 fall short of $a^2 + b^2$? Hence or otherwise calculate the projection of AC on BC.

2. ABC is an isosceles triangle in which $AB = AC$; and BE is drawn perpendicular to AC. Shew that $BC^2 = 2AC \cdot CE$.

3. In the $\triangle\ ABC$, shew that
   (i) if the $\angle\ C = 60°$, then $c^2 = a^2 + b^2 - ab$;
   (ii) if the $\angle\ C = 120°$, then $c^2 = a^2 + b^2 + ab$.

THEOREM 33.

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Let ABC be a triangle, and AX the median which bisects the base BC.

It is required to prove that

$$AB^2 + AC^2 = 2\,BX^2 + 2\,AX^2.$$

Draw AD perp. to BC; and consider the case in which AB and AC are unequal, and AD falls within the triangle.

Then of the $\angle s$ AXB, AXC, one is obtuse, and the other acute. Let the $\angle\, AXB$ be obtuse.

Then from the $\triangle\, AXB$,

$$AB^2 = BX^2 + AX^2 + 2\,BX \cdot XD. \quad \textit{Theor. } 31.$$

And from the $\triangle\, AXC$,

$$AC^2 = XC^2 + AX^2 - 2\,XC \cdot XD. \quad \textit{Theor. } 32.$$

Adding these results, and remembering that $XC = BX$, we have

$$AB^2 + AC^2 = 2\,BX^2 + 2\,AX^2. \qquad \text{Q.E.D.}$$

NOTE. The proof may easily be adapted to the case in which the perpendicular AD falls outside the triangle.

EXERCISE

In any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.

EXERCISES ON THEOREMS 31–33

1. AB is a straight line 8 cm. in length, and from its middle point O as centre with radius 5 cm. a circle is drawn; if P is any point on the circumference, shew that

$$AP^2 + BP^2 = 82 \text{ sq. cm.}$$

2. In a triangle ABC, the base BC is bisected at X. If $a = 17$ cm., $b = 15$ cm., and $c = 8$ cm., calculate the length of the median AX, and deduce the $\angle A$.

3. The base of a triangle $= 10$ cm., and the sum of the squares on the other sides $= 122$ sq. cm.; find the locus of the vertex.

4. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

The sides of a rhombus and its shorter diagonal each measure 3″; find the longer diagonal to within ·01″.

5. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 7, p. 64.]

6. $ABCD$ is a rectangle, and O any point within it: shew that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

If $AB = 6{\cdot}0''$, $BC = 2{\cdot}5''$, and $OA^2 + OC^2 = 21\frac{1}{4}$ sq. in., find the distance of O from the intersection of the diagonals.

7. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

8. In a triangle ABC, the angles at B and C are acute; if BE, CF are drawn perpendicular to AC, AB respectively, prove that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

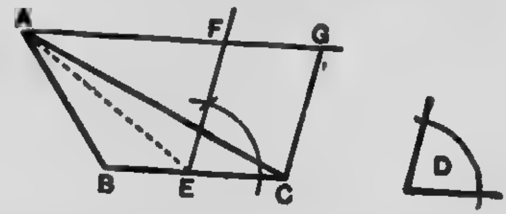
10. ABC is a triangle, and O the point of intersection of its medians: shew that

$$AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2).$$

PROBLEMS ON AREAS

Problem 17

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle



Let ABC be the given triangle, and D the given angle.

It is required to describe a parallelogram equal to ABC, and having one of its angles equal to D.

Construction. Bisect BC at E.

At E in CE, make the $\angle\ CEF$ equal to D;
through A draw AFG par^l to BC;
and through C draw CG par^l to EF.
Then $FECG$ is the required par^m.

Proof. Join AE.

Now the $\triangle$ ABE, AEC are on equal bases BE, EC, and of the same altitude;

$\therefore$ the $\triangle\ ABE$ = the $\triangle\ AEC$.
$\therefore$ the $\triangle\ ABC$ is double of the $\triangle\ AEC$.
But $FECG$ is a par^m by construction;
and it is double of the $\triangle\ AEC$,
being on the same base EC, and between the same par^{ls} EO and AG.

$\therefore$ the par^m $FECG$ = the $\triangle\ ABC$;
and one of its angles, namely CEF, = the given $\angle\ D$.

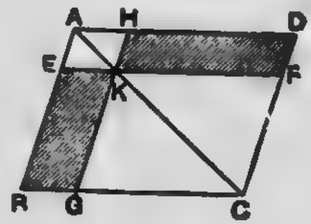
EXERCISES

(Graphical)

1. Draw a square on a side of 5 cm., and make a parallelogram of equal area on the same base, and having an angle of 45°.

Find (i) by calculation, (ii) by measurement the length of an oblique side of the parallelogram.

2. Draw any parallelogram $ABCD$ in which $AB = 2\frac{1}{2}''$ and $AD = 2''$; and on the base AB draw a rhombus of equal area.



Definition. In a parallelogram $ABCD$, if through any point K in the diagonal AC parallels EF, HG are drawn to the sides, then the figures EH, GF are called **parallelograms about** AC, and the figures EG, HF are said to be their **complements**.

3. *In the diagram of the preceding definition shew by Theorem 21 that the complements EG, HF are equal in area.*

Hence, given a parallelogram EG, and a straight line HK, deduce a construction for drawing on HK as one side a parallelogram equal and equiangular to the parallelogram EG.

4. Construct a rectangle equal in area to a given rectangle $CDEF$, and having one side equal to a given line AB.

If $AB = 6$ cm., $CD = 8$ cm., $CF = 3$ cm., find by measurement the remaining side of the constructed rectangle.

5. Given a parallelogram $ABCD$, in which $AB = 2{\cdot}4''$, $AD = 1{\cdot}8''$, and the $\angle A = 55°$. Construct a parallelogram of equal area and equiangular with $ABCD$, the greater side measuring $2{\cdot}7''$. Measure the shorter side.

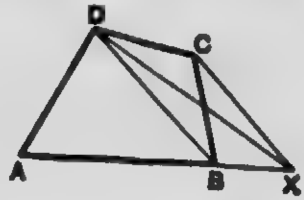
Repeat the process, giving to A any other value, and compare your results. What conclusion do you draw?

6. Draw a rectangle on a side of 5 cm. equal in area to an equilateral triangle on a side of 6 cm.

Measure the remaining side of the rectangle, and calculate its approximate area.

PROBLEM 18

To draw a triangle equal in area to a given quadrilateral.



Let $ABCD$ be the given quadrilateral.

It is required to describe a triangle equal to $ABCD$ in area.

Construction. Join DB.

Through C draw CX par^{l} to DB, meeting AB produced in X.

Join DX.

Then DAX is the required triangle.

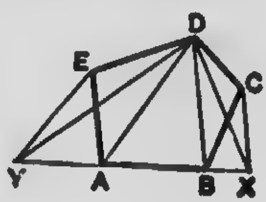
Proof. Now the ▲ XDB, CDB are on the same base DB and between the same par^{ls} DB, CX;

$\therefore$ the $\triangle XDB$ = the $\triangle CDB$ in area.

To each of these equals add the $\triangle ADB$;

then the $\triangle DAX$ = the fig. $ABCD$.

COROLLARY. In the same way it is always possible to draw a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure; and thus step by step, any rectilineal figure may be reduced to a triangle of equal area.

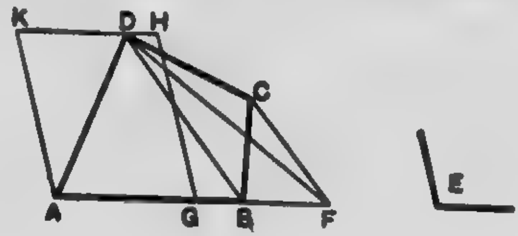


For example, in the adjoining diagram the five-sided fig. $EDCBA$ is equal in area to the four-sided fig. $EDXA$.

The fig. $EDXA$ may now be reduced to an equal $\triangle DXY$.

Problem 19

To draw a parallelogram equal in area to a given rectilineal figure, and having an angle equal to a given angle.



Let $ABCD$ be the given rectil. fig., and E the given angle.

It is required to draw a par^{m} equal to $ABCD$ and having an angle equal to E.

Construction. Join DB.

Through C draw CF par^{l} to DB, and meeting AB produced in F.

Join DF.

Then the $\triangle\ DAF$ = the fig. $ABCD$. *Prob.* 18.

Draw the par^{m} $AGHK$ equal to the $\triangle\ ADF$, and having the $\angle KAG$ equal to the $\angle E$. *Prob.* 17.

Then the par^{m} KG = the $\triangle\ ADF$

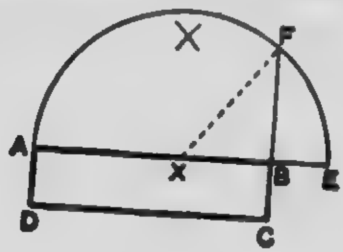
= the fig. $ABCD$;

and it has the $\angle KAG$ equal to the $\angle E$.

Note. If the given rectilineal figure has more than four sides, it must first be reduced, step by step, until it is replaced by an equivalent triangle.

Problem 20

To draw a square equal in area to a given rectangle.



Let $ABCD$ be the given rectangle.

Construction. Produce AB to E, making BE equal to BC. On AE draw a semi-circle; and produce CB to meet the circumference at F.

Then BF is a side of the required square.

Proof. Let X be the mid-point of AE, and r the radius of the semi-circle. Join XF.

Then the rect. $AC = AB \cdot BE$

$$= (r + XB)(r - XB)$$

$$= r^2 - XB^2 \qquad \text{(p. 129, E)}$$

$$= FB^2, \text{ from the rt. angled } \triangle FBX.$$

Corollary. *To describe a square equal in area to any given rectilineal figure.*

Reduce the given figure to a triangle of equal area. *Prob.* 18.
Draw a rectangle equivalent to this triangle. *Prob.* 17.
Apply to the rectangle the construction given above.

EXERCISES

(Reduction of a Rectilineal Figure to an Equivalent Triangle)

1. Draw a quadrilateral $ABCD$ from the following data:
$AB = BC = 5{\cdot}5$ cm.; $CD = DA = 4{\cdot}5$ cm.; the $\angle A = 75°$.
Reduce the quadrilateral to a triangle of equal area. Measure the base and altitude of the triangle, and hence calculate the approximate area of the given figure.

2. Draw a quadrilateral $ABCD$ having given:
$AB = 2{\cdot}8''$, $BC = 3{\cdot}2''$, $CD = 3{\cdot}3''$, $DA = 3{\cdot}6''$, and the diagonal $BD = 3{\cdot}0''$.
Construct an equivalent triangle; and hence find the approximate area of the quadrilateral.

3. On a base AB, 4 cm. in length, describe an equilateral pentagon (5 sides), having each of the angles at A and B 108°.
Reduce the figure to a triangle of equal area; and by measuring its base and altitude, calculate the approximate area of the pentagon.

4. A quadrilateral field $ABCD$ has the following measurements:
$AB = 450$ metres, $BC = 380$ m., $CD = 330$ m., $AD = 390$ m., and the diagonal $AC = 660$ m.
Draw a plan (scale 1 cm. to 50 metres). Reduce your plan to an equivalent triangle, and measure its base and altitude. Hence estimate the area of the field.

(Problems. State your construction, and give a theoretical proof.)

5. On the base of a given triangle construct a second triangle equal in area to the first and having its vertex in a given line.

6. Reduce a triangle ABC to a triangle of equal area having its base BD of given length. (D lies in BC, or BC produced.)

7. Construct a triangle equal in area to a given triangle, and having a given altitude.

8. ABC is a given triangle, and X a given point. Draw a triangle equal in area to ABC, having its vertex at X, and its base in the same straight line as BC.

9. Construct a triangle equal in area to the quadrilateral $ABCD$, having its vertex at a given point X in DC, and its base in the same straight line as AB.

10. Construct a triangle equal in area to a quadrilateral $ABCD$ and having two of its sides equal respectively to the diagonals of the quadrilateral.

11. Show how a triangle may be divided into n equal parts by straight lines drawn through one of its angular points.

12. *Bisect a triangle by a straight line drawn through a given point in one of its sides.*

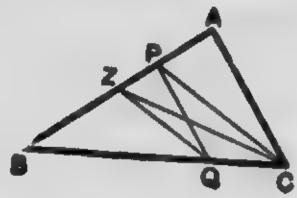
[Let ABC be the given $\triangle$, and P the given point in the side AB.

Bisect AB at Z; and join CZ, CP.

Through Z draw ZQ parallel to CP.

Join PQ.

Then PQ bisects the $\triangle$.]



13. *Trisect a triangle by straight lines drawn from a given point in one of its sides.*

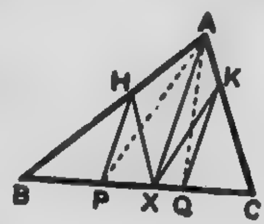
[Let ABC be the given $\triangle$, and X the given point in the side BC.

Trisect BC at the points P, Q. *Prob.* 7.

Join AX, and through P and Q draw PH and QK parallel to AX.

Join XH, XK.

These straight lines trisect the $\triangle$; as may be shewn by joining AP, AQ.]



14. Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides.

15. *Bisect a quadrilateral by a straight line drawn through an angular point.*

[Reduce the quadrilateral to a triangle of equal area, and join the vertex to the middle point of the base.]

16. Cut off from a given quadrilateral a third, a fourth, a fifth, or any part required, by a straight line drawn through a given angular point.

MISCELLANEOUS EXERCISES

1. AB and AC are unequal sides of a triangle ABC; AX is the median through A, AP bisects the angle BAC, and AD is the perpendicular from A to BC. Prove that AP is intermediate in position and magnitude to AX and AD.

2. In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle, (i) it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base; (ii) it will make with the base an angle equal to half the difference of the angles at the base.

3. In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.

4. Construct a right-angled triangle, having given the hypotenuse and the difference of the other sides.

5. Construct a triangle, having given the base, the difference of the angles at the base, and (i) the difference, (ii) the sum, of the remaining sides.

6. Construct an isosceles triangle, having given the base and the sum of one of the equal sides and the perpendicular from the vertex to the base.

7. Shew how to divide a given straight line so that the square on one part may be double the square on the other.

8. $ABCD$ is a parallelogram, and O is any point without the angle BAD or its opposite vertical angle; shew that the triangle OAC is equal to the sum of the triangles OAD, OAB.

If O is within the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the difference of the triangles OAD, OAB.

9. Find the locus of the intersection of the medians of triangles described on a given base and of given area.

10. On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line.

11. $ABCD$ is a parallelogram made of rods connected by hinges. If AB is fixed, find the locus of the middle point of CD.

PART III

THE CIRCLE

Definitions and First Principles

1. A **circle** is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.

The fixed point is called the **centre**, and the bounding line is called the **circumference**.

Note. According to this definition the term circle strictly applies to the *figure* contained by the circumference; it is often used, however, for the circumference itself when no confusion is likely to arise.

2. A **radius** of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal.

3. A **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

4. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

It will be proved on page 146 that a diameter divides a circle into two identically equal parts.

5. Circles that have the same centre are said to be **concentric**.

From these definitions we draw the following inferences :

(i) A circle is a *closed* curve; so that if the circumference is crossed by a straight line, this line if produced will cross the circumference at a second point.

(ii) The distance of a point from the centre of a circle is greater or less than the radius according as the point is without or within the circumference.

(iii) A point is outside or inside a circle according as its distance from the centre is greater or less than the radius.

(iv) Circles of equal radii are identically equal. For by superposition of one centre on the other the circumferences must coincide at every point.

(v) Concentric circles of unequal radii cannot intersect, for the distance from the centre of every point on the smaller circle is less than the radius of the larger.

(vi) If the circumferences of two circles have a common point they cannot have the same centre, unless they coincide altogether.

6. An **arc** of a circle is any part of the circumference.

7. A **chord** of a circle is a straight line joining any two points on the circumference.

NOTE. From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc**. Thus the major arc is *greater*, and the minor arc *less* than the semi-circumference.

The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another.

Symmetry

Some elementary properties of circles are easily proved by considerations of symmetry. For convenience the definition given previously is here repeated.

Definition 1. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry.**

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

Definition 2. Let AB be a straight line and P a point outside it.

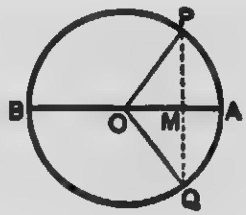
From P draw PM perp. to AB, and produce it to Q, making MQ equal to PM.

Then if the figure is folded about AB, the point P may be made to coincide with Q, for the $\angle AMP =$ the $\angle AMQ$ and $MP = MQ$.

The points P and Q are said to be **symmetrically opposite** with regard to the axis AB, and each point is said to be the **image** of the other in the axis.

Note. A point and its image are equidistant from every point on the axis. See Prob. 14, page 91.

A circle is symmetrical about any diameter.



Let $APBQ$ be a circle of which O is the centre, and AB any diameter.

It is required to prove that the circle is symmetrical about AB.

Proof. Let OP and OQ be two radii making any equal $\angle s$ AOP, AOQ on opposite sides of OA.

Then if the figure is folded about AB, OP may be made to fall along OQ, since the $\angle AOP =$ the $\angle AOQ$.

And thus P will coincide with Q, since $OP = OQ$.

Thus every point in the arc APB must coincide with some point in the arc AQB; that is, the two parts of the circumference on each side of AB can be made to coincide.

$\therefore$ the circle is symmetrical about the diameter AB.

COROLLARY. If PQ is drawn cutting AB at M, then on folding the figure about AB, since P falls on Q, MP will coincide with MQ,

$$\therefore MP = MQ;$$

and the $\angle OMP$ will coincide with the $\angle OMQ$;

$\therefore$ these angles, being adjacent, are rt. $\angle s$;

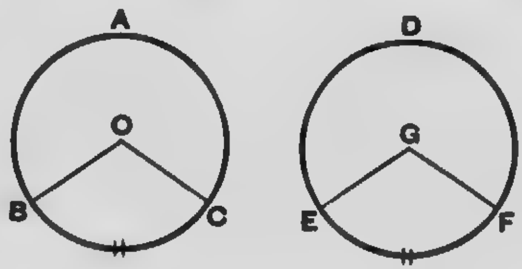
$\therefore$ the points P and Q are symmetrically opposite with regard to AB.

Hence, conversely, *if a circle passes through a given point* P, *it also passes through the symmetrically opposite point with regard to any diameter.*

SOME PROPERTIES OF EQUAL CIRCLES

The student should prove for himself the following properties of equal circles. A, B, D, and E may readily be proven by superposition, while C is a simple exercise on Theorem 7.

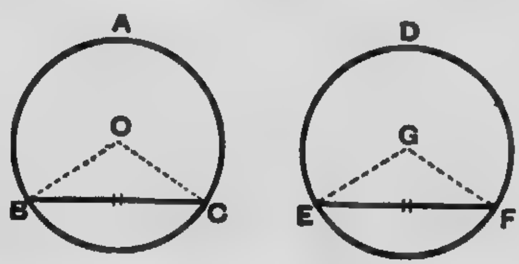
A. *In equal circles angles at the centre which stand on equal arcs are equal.*



B. *In equal circles arcs which subtend equal angles at the centre are equal.*

C. *In equal circles arcs which are cut off by equal chords are equal, the major arc to the major and the minor arc to the minor.*

(a) Prove $\angle BOC = \angle EGF$, *Th.* 7.
(b) Hence arc BC = arc EF, *Th.* 2.



D. *In equal circles chords which cut off equal arcs are equal.*

E. *In equal circles sectors* (see Def. p. 161) *which have equal angles are equal.*

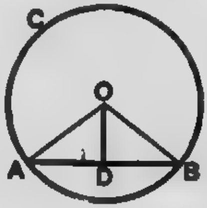
Note. State and prove these properties for the *same* circle.

ON CHORDS

Theorem 34. [Euclid III. 3]

If a straight line drawn from the centre of a circle bisects a chord which does not pass through the centre, it cuts the chord at right angles.

Conversely, if it cuts the chord at right angles, it bisects it.



Let ABC be a circle whose centre is O; and let OD bisect a chord AB which does not pass through the centre.

It is required to prove that OD is perp. to AB.

Join OA, OB.

Proof. Then in the $\triangle$ ADO, BDO,

because $\begin{cases} AD = BD, \text{ by hypothesis,} \\ OD \text{ is common,} \\ \text{and } OA = OB, \text{ being radii of the circle;} \end{cases}$

$\therefore$ the $\angle\ ADO$ = the $\angle\ BDO$, *Theor.* 7.

and these are adjacent angles;

$\therefore$ OD is perp. to AB. Q.E.D.

Conversely. Let OD be perp. to the chord AB.

It is required to prove that OD bisects AB.

Proof. In the $\triangle$ ODA, ODB,

because $\begin{cases} \text{the } \angle s\ ODA, ODB \text{ are \textit{right angles},} \\ \text{the hypotenuse } OA = \text{the hypotenuse } OB, \\ \text{and } OD \text{ is common;} \end{cases}$

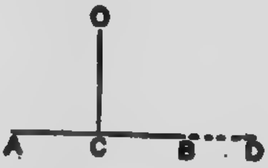
$\therefore$ $DA = DB$; *Theor.* 18.

that is, OD bisects AB at D. Q.E.D.

Corollary 1. *The straight line which bisects a chord at right angles passes through the centre.*

Corollary 2. *A straight line cannot meet a circle at more than two points.*

For suppose a st. line meets a circle whose centre is O at the points A and B.



Draw OC perp. to AB.

Then $AC = CB$.

Now if the circle were to cut AB in a third point D, AC would also be equal to CD, which is impossible.

Corollary 3. *A chord of a circle lies wholly within it.*

EXERCISES

(Numerical and Graphical)

1. In the figure of Theorem 34, if $AB = 8$ cm., and $OD = 3$ cm., find OB. Draw the figure, and verify your result by measurement.

2. Calculate the length of a chord which stands at a distance 5″ from the centre of a circle whose radius is 13″.

3. In a circle of 1″ radius draw two chords 1·6″ and 1·2″ in length. Calculate and measure the distance of each from the centre.

4. Draw a circle whose diameter is 8·0 cm. and place in it a chord 6·0 cm. in length. Calculate to the nearest millimetre the distance of the chord from the centre; and verify your result by measurement.

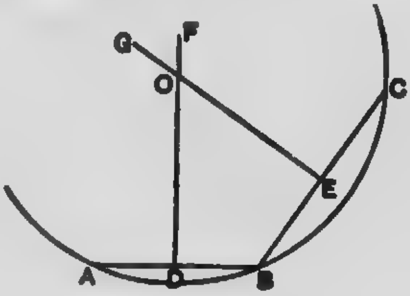
5. Find the distance from the centre to a chord 5 ft. 10 in. in length in a circle whose diameter is 2 yds. 2 in. Verify the result graphically by drawing a figure in which 1 cm. represents 10″.

6. AB is a chord 2·4″ long in a circle whose centre is O and whose radius is 1·3″; find the area of the triangle OAB in square inches.

7. Two points P and Q are 3″ apart. Draw a circle with radius 1·7″ to pass through P and Q. Calculate the distance of its centre from the chord PQ, and verify by measurement.

Theorem 35

One circle, and only one, can pass through any three points not in the same straight line.



Let A, B, C be three points not in the same straight line.

It is required to prove that one circle, and only one, can pass through A, B, and C.

Join AB, BC.

Let AB and BC be bisected at right angles by the lines DF, EG.

Then since AB and BC are not in the same st. line, DF and EG are not par[l].

Let DF and EG meet in O.

Proof. Because DF bisects AB at right angles,

$\therefore$ every point on DF is equidistant from A and B.

Prob. 14.

Similarly every point on EG is equidistant from B and C.

$\therefore$ O, the only point common to DF and EG, is equidistant from A, B, and C;

and there is no other point equidistant from A, B, and C.

$\therefore$ a circle having its centre at O and radius OA will pass through B and C; and this is the only circle which will pass through the three given points. Q.E.D.

COROLLARY 1. *The size and position of a circle are fully determined if three of its points are known;* for then the position of the centre and length of the radius can be found.

COROLLARY 2. *Two circles cannot cut one another in more than two points without coinciding entirely;* for if they cut at three points they would have the same centre and radius.

HYPOTHETICAL CONSTRUCTION. *From Theorem 35 it appears that we may suppose a circle to be drawn through any three points not in the same straight line.*

Thus, one circle passes through the vertices of any triangle.

DEFINITION. The circle passing through the vertices of a triangle is said to be **circumscribed** about the triangle. The circle, its centre, and its radius are called the **circum-circle**, the **circum-centre**, and the **circum-radius** of the triangle.

EXERCISES ON THEOREMS 34 AND 35

(*Theoretical*)

1. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

2. Two circles, whose centres are at A and B, intersect at C, D; and M is the middle point of the common chord. Shew that AM and BM are in the same straight line.

Hence prove that the line of centres bisects the common chord at right angles.

3. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

4. *Find the locus of the centres of all circles which pass through two given points.*

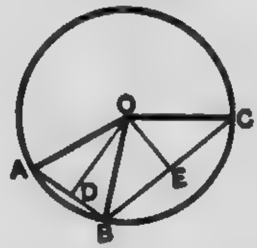
5. *Describe a circle that shall pass through two given points and have its centre in a given straight line.*

When is this impossible?

6. *Describe a circle of given radius to pass through two given points.*
When is this impossible?

*THEOREM 36. [Euclid III. 9]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and O a point within it from which more than two equal st. lines are drawn to the $\bigcirc^{ce}$, namely OA, OB, OC.

It is required to prove that O is the centre of the circle ABC.

Join AB, BC.

Let D and E be the middle points of AB and BC respectively.

Join OD, OE.

Proof. In the $\triangle^s$ ODA, ODB,

because $\begin{cases} DA = DB, \\ DO \text{ is common,} \\ \text{and } OA = OB, \text{ by hypothesis;} \end{cases}$

$\therefore$ the $\angle ODA =$ the $\angle ODB$; *Theor.* 7.

$\therefore$ these angles, being adjacent, are rt. $\angle^s$.

Hence DO bisects the chord AB at right angles, and therefore passes through the centre. *Theor.* 34, *Cor.* 1.

Similarly it may be shewn that EO passes through the centre.

$\therefore$ O, which is the only point common to DO and EO, must be the centre. Q.E.D.

EXERCISES ON CHORDS

(Numerical and Graphical)

1. *AB* and *BC* are lines at right angles, and their lengths are 1·6″ and 3·0″ respectively. Draw the circle through the points *A*, *B*, and *C*; find the length of its radius, and verify your result by measurement.

2. Draw a circle in which a chord 6 cm. in length stands at a distance of 3 cm. from the centre.

Calculate (to the nearest millimetre) the length of the radius, and verify your result by measurement.

3. Draw a circle on a diameter of 8 cm., and place in it a chord equal to the radius.

Calculate (to the nearest millimetre) the distance of the chord from the centre, and verify by measurement.

4. Two circles, whose radii are respectively 26 inches and 25 inches, intersect at two points which are 4 feet apart. Find the distance between their centres.

Draw the figure (scale 1 cm. to 10″), and verify your result by measurement.

5. Two parallel chords of a circle whose diameter is 13″ are respectively 5″ and 12″ in length; shew that the distance between them is either 8·5″ or 3·5″.

6. Two parallel chords of a circle on the same side of the centre are 6 cm. and 8 cm. in length respectively, and the perpendicular distance between them is 1 cm. Calculate and measure the radius.

(Theoretical)

7. The line joining the middle points of two parallel chords of a circle passes through the centre.

8. *Find the locus of the middle points of parallel chords in a circle.*

9. Two intersecting chords of a circle cannot bisect each other unless each is a diameter.

10. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

11. Shew that rectangles are the only parallelograms that can be inscribed in a circle.

THEOREM 37. [Euclid III. 14]

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.

Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

First. Let $AB = CD$.

It is required to prove that AB and CD are equidistant from O.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

$\therefore$ OF bisects AB; *Theor.* 34.

$\therefore$ AF is half of AB.

Similarly CG is half of CD.

But, by hypothesis, $AB = CD$,

$\therefore$ $AF = CG$.

Now in the $\triangle$s OFA, OGC,

because $\begin{cases} \text{the } \angle\text{s } OFA, OGC \text{ are } \textit{right angles}, \\ \text{the hypotenuse } OA = \text{the hypotenuse } OC, \\ \text{and } AF = CG; \end{cases}$

$\therefore$ the triangles are equal in all respects; *Theor.* 18.

so that $OF = OG$;

that is, AB and CD are equidistant from O.

Q.E.D.

Conversely. Let $OF = OG$.

It is required to prove that $AB = CD$.

Proof. As before it may be shewn that AF is half of AB, and CG half of CD.

Then in the ▲ OFA, OGC,

because { the ∠ OFA, OGC are *right angles*,
the hypotenuse OA = the hypotenuse OC,
and $OF = OG$;

∴ $AF = CG$; *Theor.* 18.

∴ the doubles of these are equal;

that is, $AB = CD$. Q.E.D.

EXERCISES

(Theoretical)

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

3. If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is the same for all positions of AB. [See Ex. 9, p. 64.]

(Graphical)

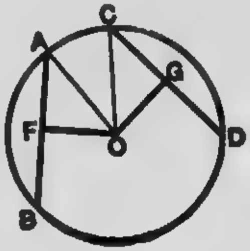
6. In a circle of radius 4·1 cm. any number of chords are drawn each 1·8 cm. in length. Shew that the middle points of these chords all lie on a circle. Calculate and measure the length of its radius, and draw the circle.

7. The centres of two circles are 4″ apart, their common chord is 2·4″ in length, and the radius of the larger circle is 3·7″. Give a construction for finding the points of intersection of the two circles, and find the radius of the smaller circle.

THEOREM 38. [Euclid III. 15]

Of any two chords of a circle, that which is nearer to the centre is greater than one more remote.

Conversely, the greater of two chords is nearer to the centre than the less.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

It is required to prove that

(i) *if OF is less than OG, then AB is greater than CD;*

(ii) *if AB is greater than CD, then OF is less than OG.*

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

$\therefore$ OF bisects AB;

$\therefore$ AF is half of AB.

Similarly CG is half of CD.

Now $OA = OC$;

$\therefore$ the sq. on OA = the sq. on OC.

But since the $\angle OFA$ is a rt. angle,

$\therefore$ the sq. on OA = the sqq. on OF, FA.

Similarly the sq. on OC = the sqq. on OG, GC.

$\therefore$ the sqq. on OF, FA = the sqq. on OG, GC.

(i) Hence if OF is given less than OG,
the sq. on OF is less than the sq. on OG.
$\therefore$ the sq. on FA is greater than the sq. on GC;
$\therefore$ FA is greater than GC;
$\therefore$ AB is greater than CD.

(ii) But if AB is given greater than CD,
that is, if FA is greater than GC;
then the sq. on FA is greater than the sq. on GC.
$\therefore$ the sq. on OF is less than the sq. on OG;
$\therefore$ OF is less than OG. Q.E.D.

COROLLARY. *The greatest chord in a circle is a diameter.*

EXERCISES

(Miscellaneous)

1. Through a given point within a circle draw the least possible chord.

2. Draw a triangle ABC in which $a = 3\cdot5''$, $b = 1\cdot2''$, $c = 3\cdot7''$. Through the ends of the side a draw a circle with its centre on the side c. Calculate and measure the radius.

3. Draw the circum-circle of a triangle whose sides are $2\cdot6''$, $2\cdot8''$, and $3\cdot0''$. Measure its radius.

4. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB; what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

5. Describe the change of direction of the chord XY (in Ex. 4) as Z moves from one end of AB to its middle point.

6. What direction does XY take when Z reaches the middle point of AB?

7. Consider the position of XY when Z gets very near to A.

NOTE. For exercises on Theorems 34–38, see page 160.

ON ANGLES IN SEGMENTS, AND ANGLES AT THE CENTRES AND CIRCUMFERENCES OF CIRCLES

THEOREM 39. [Euclid III. 20]

The angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

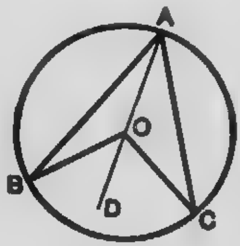


Fig. 1.

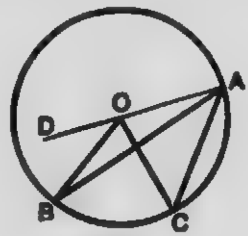


Fig. 2.

Let ABC be a circle, of which O is the centre; and let BOC be the angle at the centre, and BAC an angle at the $\bigcirc^{ce}$ standing on the same arc BC.

It is required to prove that the $\angle BOC$ *is twice the* $\angle BAC$.

Join AO, and produce it to D.

Proof. In the $\triangle OAB$, because $OB = OA$,

$\therefore$ the $\angle OAB =$ the $\angle OBA$.

$\therefore$ the sum of the $\angle$s $OAB, OBA =$ twice the $\angle OAB$.

But the ext. $\angle BOD =$ the sum of the $\angle$s OAB, OBA;

$\therefore$ the $\angle BOD =$ twice the $\angle OAB$.

Similarly the $\angle DOC =$ twice the $\angle OAC$.

$\therefore$, adding these results in Fig. 1, and taking the difference in Fig. 2, it follows in each case that

the $\angle BOC =$ twice the $\angle BAC$. Q.E.D.

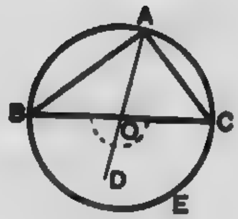


Fig. 3.

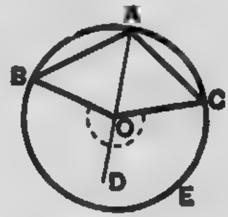


Fig. 4.

Obs. If the arc BEC, on which the angles stand, is a semi-circumference, as in Fig. 3, the $\angle\ BOC$ at the centre is a *straight angle;* and if the arc BEC is greater than a semi-circumference, as in Fig. 4, the $\angle\ BOC$ at the centre is *reflex*. But the proof for Fig. 1 applies without change to both these cases, shewing that whether the given arc is greater than, equal to, or less than a semi-circumference,

the $\angle\ BOC$ = *twice the* $\angle\ BAC$, *on the same arc* BEC.

DEFINITIONS

A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.

NOTE. The chord of a segment is sometimes called its base.

An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

We have seen in Theorem 35 that a circle may be drawn through *any three* points not in a straight line. But it is only under certain conditions that a circle can be drawn through more than three points.

DEFINITION. If four or more points are so placed that a circle may be drawn through them, they are said to be **concyclic.**

EXERCISES

(*Miscellaneous*)

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

2. If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

3. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles are equal.

4. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal.

5. Two circles of diameters 74 and 40 inches have a common chord 2 feet in length; find the distance between their centres.

6. Draw two circles of radii 1·0″ and 1·7″, and with their centres 2·1″ apart. Find by calculation, and by measurement, the length of the common chord, and its distance from the two centres.

7. Find the greatest and least straight lines which have one extremity on each of two given non-intersecting circles.

8. If from any point on the circumference of a circle straight lines are drawn to the circumference, the greatest is that which passes through the centre; and of any two such lines the greater is that which subtends the greater angle at the centre.

9. Of all straight lines drawn through a point of intersection of two circles and terminated by the circumferences, the greatest is that which is parallel to the line of centres.

10. *If from any internal point, not the centre, straight lines are drawn to the circumference of a circle, then the greatest is that which passes through the centre, and the least is the remaining part of that diameter; and of any other two such lines the greater is that which subtends the greater angle at the centre.*

11. *If from any external point straight lines are drawn to the circumference of a circle, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre; and of any other two such lines, the greater is that which subtends the greater angle at the centre.*

EXERCISES ON THEOREM 39

1. *Prove that the angle in a semi-circle is a right angle.*

2. *The angle in a segment of a circle greater than a semi-circle is an acute angle.*

3. *The angle in a segment of a circle less than a semi-circle is an obtuse angle.*

4. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

5. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle; shew that the points P, B, Q are collinear.

6. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

7. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

8. A straight rod of given length slides between two straight rulers placed at right angles to one another; find the locus of its middle point.

9. *Find the locus of the middle points of chords of a circle drawn through a fixed point. Distinguish between the cases when the given point is within, on, or without the circumference.*

10. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*

11. *If two chords intersect without a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

12. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

DEFINITION. A **sector** of a circle is a figure bounded by two radii and the arc intercepted between them.

THEOREM 40. [Euclid III. 21]

Angles in the same segment of a circle are equal.

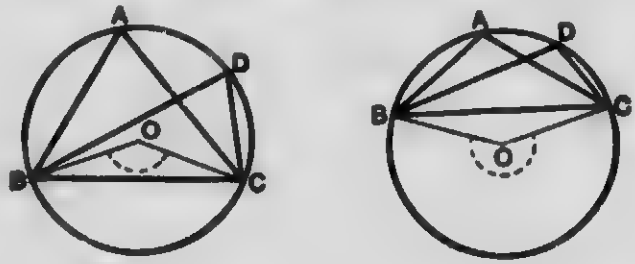


Fig. 1. Fig. 2.

Let BAC, BDC be angles in the same segment $BADC$ of a circle, whose centre is O.

It is required to prove that the $\angle BAC =$ *the* $\angle BDC$.

Join BO, OC.

Proof. Because the $\angle BOC$ is at the centre, and the $\angle BAC$ at the $\bigcirc^{ce}$, standing on the same arc BC,

$\therefore$ the $\angle BOC =$ twice the $\angle BAC$. *Theor.* 39.

Similarly the $\angle BOC =$ twice the $\angle BDC$.

$\therefore$ the $\angle BAC =$ the $\angle BDC$. Q.E.D.

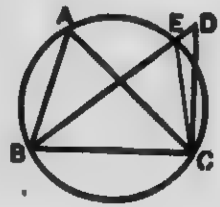
NOTE. The given segment may be greater than a semi-circle as in Fig. 1, or less than a semi-circle as in Fig. 2; in the latter case the angle BOC will be reflex. But by virtue of the extension of Theorem 39 given on page 159, the above proof applies equally to both figures.

CONVERSE OF THEOREM 40

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.

Let BAC, BDC be two equal angles standing on the same base BC, and on the same side of it.

It is required to prove that A and D lie on an arc of a circle having BC as its chord.



Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts BD or BD produced at the point E.

Join EC.

Proof. Then the $\angle BAC$ = the $\angle BEC$ in the same segment.

But, by hypothesis, the $\angle BAC$ = the $\angle BDC$;

$\therefore$ the $\angle BEC$ = the $\angle BDC$;

which is impossible unless E coincides with D;

$\therefore$ the circle through B, A, C must pass through D.

COROLLARY. *The locus of the vertices of triangles drawn on the same side of a given base, and with equal vertical angles, is an arc of a circle.*

EXERCISES ON THEOREM 40

1. In Fig. 1, if the angle BDC is 74°, find the number of degrees in each of the angles BAC, BOC, OBC.

2. In Fig. 2, let BD and CA intersect at X. If the angle DXC = 40°, and the angle XCD = 25°, find the number of degrees in the angle BAC and in the reflex angle BOC.

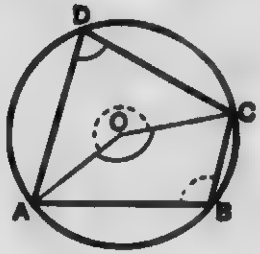
3. In Fig. 1, if the angles CBD, BCD are respectively 43° and 82°, find the number of degrees in the angles BAC, OBD, OCD.

4. Shew that in Fig. 2 the angle OBC is always less than the angle BAC by a right angle.

[For further Exercises on Theorem 40 see page 166.]

THEOREM 41. [Euclid III. 22]

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let $ABCD$ be a quadrilateral inscribed in the $\odot\ ABC$.

It is required to prove that

(i) *the* $\angle s\ ADC, ABC$ *together = two rt. angles.*

(ii) *the* $\angle s\ BAD, BCD$ *together = two rt. angles.*

Suppose O is the centre of the circle.

Join OA, OC.

Proof. Since the $\angle\ ADC$ at the $\bigcirc^{ce}$ = half the $\angle\ AOC$ at the centre, standing on the same arc ABC;

and the $\angle\ ABC$ at the $\bigcirc^{ce}$ = half the reflex $\angle\ AOC$ at the centre, standing on the same arc ADC;

$\therefore$ the $\angle s\ ADC, ABC$ together = half the sum of the $\angle\ AOC$ and the reflex $\angle\ AOC$.

But the latter angles make up four rt. angles.

$\therefore$ the $\angle s\ ADC, ABC$ together = two rt. angles.

Similarly the $\angle s\ BAD, BCD$ together = two rt. angles.

Q.E.D.

NOTE. The results of Theorems 40 and 41 should be carefully compared. From Theorem 40 we learn that angles in the *same* segment are *equal*. From Theorem 41 we learn that angles in *conjugate* segments are *supplementary*.

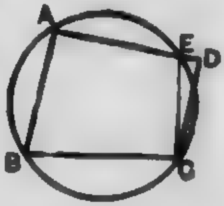
DEFINITION. A quadrilateral is called **cyclic** when a circle can be drawn through its four vertices.

CONVERSE OF THEOREM 41

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

Let $ABCD$ be a quadrilateral in which the opposite angles at B and D are supplementary.

It is required to prove that the points A, B, C, D are concyclic.



Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts AD or AD produced in the point E.

Join EC.

Proof. Then since $ABCE$ is a cyclic quadrilateral,

$\therefore$ the $\angle AEC$ is the supplement of the $\angle ABC$.

But, by hypothesis, the $\angle ADC$ is the supplement of the $\angle ABC$;

$\therefore$ the $\angle AEC$ = the $\angle ADC$;

which is impossible unless E coincides with D.

$\therefore$ the circle which passes through A, B, C must pass through D;

that is, A, B, C, D are concyclic. Q.E.D.

EXERCISES ON THEOREM 41

1. In a circle of 1·6″ radius inscribe a quadrilateral $ABCD$, making the angle ABC equal to 126°. Measure the remaining angles, and hence verify in this case that opposite angles are supplementary.

2. Prove Theorem 41 by the aid of Theorems 40 and 16, after first joining the opposite vertices of the quadrilateral.

3. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

4. ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y; shew that the four points B, C, X, Y lie on a circle.

5. *If one side of a cyclic quadrilateral is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

EXERCISES ON ANGLES IN A CIRCLE

1. P is any point on the arc of a segment of which AB is the chord; shew that the sum of the angles PAB, PBA is constant.

2. PQ and RS are two chords of a circle intersecting at X; prove that the triangles PXS, RXQ are equiangular to one another.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences; shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences; shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB; and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

6. *If AB is a fixed chord of a circle and P any point on one of the arcs cut off by it, then the bisector of the angle APB cuts the conjugate arc in the same point for all positions of P.*

7. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them; if PQ is joined, cutting AB in X and AC in Y, shew that $AX = AY$.

8. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Shew that the angles of the triangle XYZ are respectively

$$90^\circ - \frac{A}{2}, \quad 90^\circ - \frac{B}{2}, \quad 90^\circ - \frac{C}{2}.$$

9. Two circles intersect at A and B; and through these points lines are drawn from any point P on the circumference of one of the circles; shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

10. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

11. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn; shew that the chord PX is equal to the chord QY.

12. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences; shew that the straight lines which join their extremities towards the same parts are equal.

13. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences; shew that $BP = BQ$.

14. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure $BXAYC$ must have four of its sides equal.

What relation must subsist among the angles of the triangle ABC, in order that the figure $BXAYC$ may be equilateral?

15. $ABCD$ is a cyclic quadrilateral, and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q; if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

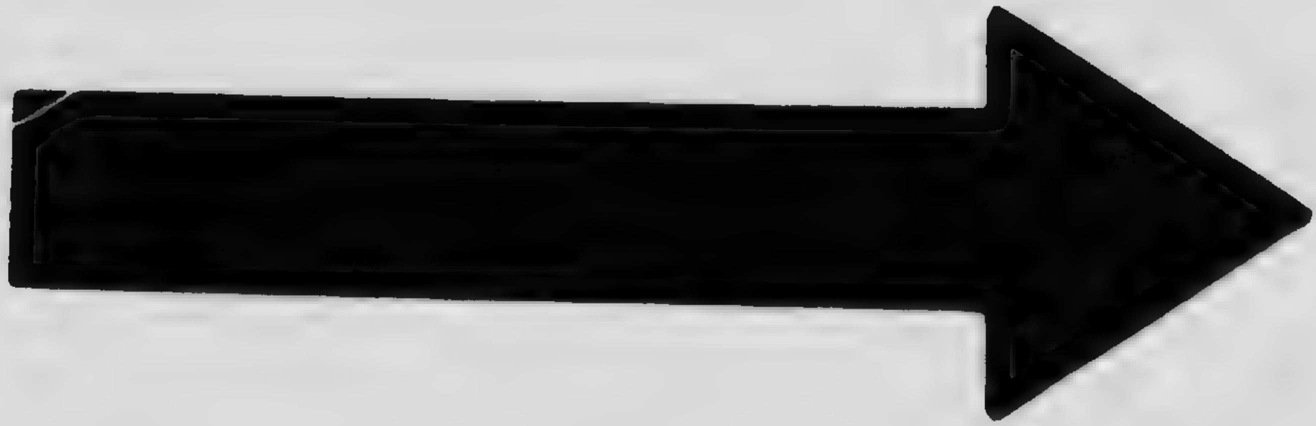
16. *P, Q, R are the middle points of the sides of a triangle, and X is the foot of the perpendicular let fall from one vertex on the opposite side; shew that the four points P, Q, R, X are concyclic.*

[See page 64, Ex. 2; also Prob. 10, p. 83.]

17. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

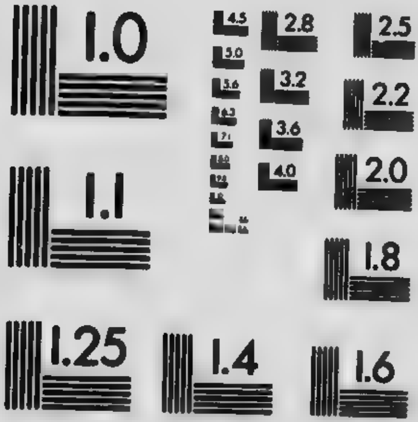
18. If a series of triangles are drawn standing on a fixed base and having a given vertical angle, shew that the bisectors of the vertical angles all pass through a fixed point.

19. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A; if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C.



MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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TANGENCY

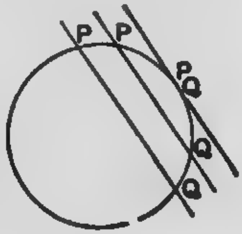
Definitions and First Principles

1. A **secant** of a circle is a straight line of indefinite length which cuts the circumference at two points.

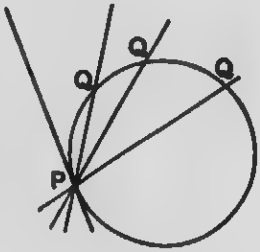
2. If a secant moves in such a way that the two points in which it cuts the circle continually approach one another, then in the ultimate position when these two points become one, the secant becomes a **tangent** to the circle, and is said to **touch** it at the point at which the two intersections coincide. This point is called the **point of contact**.

For instance:

(i) Let a secant cut the circle at the points P and Q, and suppose it to recede from the centre, moving so as to be always parallel to its original position; then the two points P and Q will clearly approach one another and finally coincide. In the ultimate position when P and Q become one point, the straight line becomes a tangent to the circle at that point.



(ii) Let a secant cut the circle at the points P and Q, and suppose it to be turned about the point P so that while P remains fixed, Q moves on the circumference nearer and nearer to P. Then the line PQ in its ultimate position, when Q coincides with P, is a tangent at the point P.



Since a secant can cut a circle at *two* points only, it is clear that a tangent can have only one point in common with the circumference, namely the point of contact, at which two points of section coincide. Hence we may define a tangent as follows:

3. A **tangent** to a circle is a straight line which meets the circumference at one point only; and though produced indefinitely does not cut the circumference.

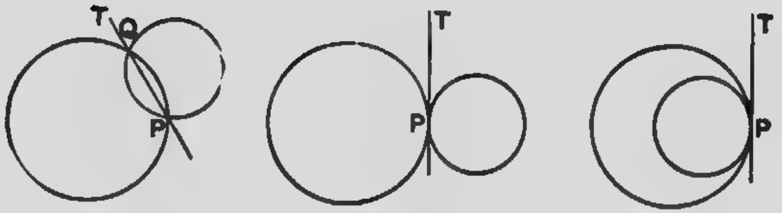


Fig. 1. Fig. 2. Fig. 3.

4. Let two circles intersect (as in Fig. 1) in the points P and Q, and let one of the circles turn about the point P, which remains fixed, in such a way that Q continually approaches P. Then in the ultimate position, when Q coincides with P (as in Figs. 2 and 3), the circles are said to **touch** one another at P.

Since two circles cannot intersect in more than *two* points, two circles which touch one another cannot have more than *one* point in common, namely the point of contact at which the two points of section coincide. Hence circles are said to **touch** one another when they meet, but do not cut one another.

NOTE. When each of the circles which meet is *outside the other*, as in Fig. 2, they are said to touch one another **externally**, or to have **external contact**; when one of the circles is *within the other*, as in Fig. 3, the first is said to touch the other **internally**, or to have **internal contact** with it.

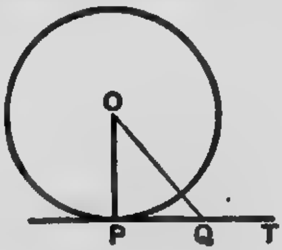
INFERENCE FROM DEFINITIONS 2 AND 4

If in Fig. 1, TQP is a common chord of two circles one of which is made to turn about P, then when Q is brought into coincidence with P, the line TP passes through two coincident points on each circle, as in Figs. 2 and 3, and therefore becomes a tangent to each circle. Hence

Two circles which touch one another have a common tangent at their point of contact.

THEOREM 42

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



Let PT be a tangent at the point P to a circle whose centre is O.

It is required to prove that PT is perpendicular to the radius OP.

Proof. Take *any* point Q in PT, and join OQ.

Then since PT is a tangent, every point in it except P is outside the circle.

$\therefore$ OQ is greater than the radius OP.

And this is true for every point Q in PT;

$\therefore$ OP is the shortest distance from O to PT.

Hence OP is perp. to PT. *Theor.* 12, *Cor.* 1.

Q.E.D.

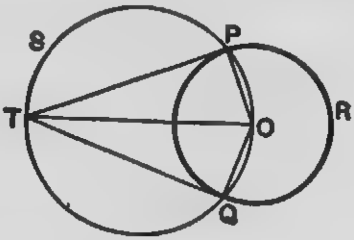
COROLLARY 1. Since there can be only one perpendicular to OP at the point P, it follows that *one and only one tangent can be drawn to a circle at a given point on the circumference.*

COROLLARY 2. Since there can be only one perpendicular to PT at the point P, it follows that *the perpendicular to a tangent at its point of contact passes through the centre.*

COROLLARY 3. Since there can be only one perpendicular from O to the line PT, it follows that *the radius drawn perpendicular to the tangent passes through the point of contact.*

THEOREM 43

Two tangents can be drawn to a circle from an external point.



Let PQR be a circle whose centre is O, and let T be an external point.

It is required to prove that there can be two tangents drawn to the circle from T.

Join OT, and let TSO be the circle on OT as diameter.

This circle will cut the $\odot$ PQR in two points, since T is without, and O is within, the $\odot$ PQR. Let P and Q be these points.

Join TP, TQ; OP, OQ.

Proof. Now each of the $\angle$s TPO, TQO, being in a semicircle, is a rt. angle;

$\therefore$ TP, TQ are perp. to the radii OP, OQ respectively.

$\therefore$ TP, TQ are tangents at P and Q. *Theor.* 42.

Q.E.D.

COROLLARY. *The two tangents to a circle from an external point are equal, and subtend equal angles at the centre.*

For in the $\triangle$s TPO, TQO,

because $\begin{cases} \text{the } \angle\text{s } TPO, TQO \text{ are } \textit{right angles}, \\ \text{the hypotenuse } TO \text{ is common,} \\ \text{and } OP = OQ, \text{ being radii;} \end{cases}$

$\therefore$ $TP = TQ$,

and the $\angle$ TOP = the $\angle$ TOQ. *Theor.* 18.

EXERCISES ON THE TANGENT

(Numerical and Graphical)

1. Draw two concentric circles with radii 5·0 cm. and 3·0 cm. Draw a series of chords of the former to touch the latter. Calculate and measure their lengths, and account for their being equal.

2. In a circle of radius 1·0″ draw a number of chords each 1·6″ in length. Shew that they all touch a concentric circle, and find its radius.

3. Find to the nearest millimetre the length of any chord of a circle of radius 5·0 cm., which touches a concentric circle of radius 2·5 cm., and check your work by measurement.

4. In the figure of Theorem 43, if $OP = 5''$, $TO = 13''$, find the length of TP and TQ. Draw the figure (scale 2 cm. to 5″), and measure to the nearest degree the angles subtended at O by the tangents.

5. The tangents from T to a circle whose radius is 0·7″ are each 2·4″ in length. Find the distance of T from the centre of the circle. Draw the figure and check your result graphically.

(Theoretical)

6. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

7. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

8. If PQ is joined in the figure of Theorem 43 shew that the angle PTQ is double the angle OPQ.

9. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

10. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

11. *Find the locus of the centres of all circles which touch (i) a given straight line at a given point, (ii) each of two parallel straight lines, (iii) each of two intersecting straight lines.*

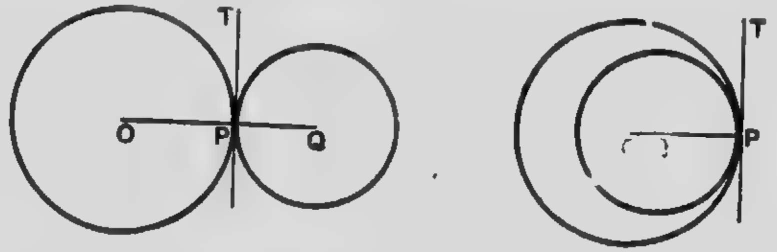
12. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

State and prove the converse theorem.

13. If a quadrilateral is described about a circle, the angles subtended at the centre by any two opposite sides are supplementary.

THEOREM 44

If two circles touch one another, the centres and the point of contact are in one straight line.



Let two circles whose centres are O and Q touch at the point P.

It is required to prove that O, P, and Q are in one straight line.

Join OP, QP.

Proof. Since the given circles touch at P, they have a common tangent at that point. *Page* 169.

Suppose PT to touch both circles at P.

Then since OP and QP are radii drawn to the point of contact,

$\therefore$ OP and QP are both perp. to PT ;

$\therefore$ OP and QP are in one st. line. *Theor.* 2.

That is, the points O, P, and Q are in one st. line. Q.E.D.

COROLLARIES. (i) *If two circles touch externally the distance between their centres is equal to the sum of their radii.*

(ii) *If two circles touch internally, the distance between their centres is equal to the difference of their radii.*

EXERCISES ON THE CONTACT OF CIRCLES

(Numerical and Graphical)

1. From centres 2·6″ apart draw two circles with radii 1·7″ and 0·9″ respectively. Why and where do these circles touch?

If circles of the above radii are drawn from centres 0·8″ apart, prove that they touch. How and why does the contact differ from that in the former case?

2. Draw a triangle ABC in which $a = 8$ cm., $b = 7$ cm., and $c = 6$ cm. From A, B, and C as centres draw circles of radii 2·5 cm., 3·5 cm., and 4·5 cm. respectively; and shew that these circles touch in pairs.

3. In the triangle ABC, right-angled at C, $a = 8$ cm. and $b = 6$ cm.; and from centre A with radius 7 cm. a circle is drawn. Find the radius of a circle drawn from centre B to touch the first circle.

4. A and B are the centres of two fixed circles which touch internally. If P is the centre of any circle which touches the larger circle internally and the smaller externally, prove that $AP + BP$ is constant.

If the fixed circles have radii 5·0 cm. and 3·0 cm. respectively, verify the general result by taking different positions for P.

5. AB is a line 4″ in length, and C is its middle point. On AB, AC, CB semi-circles are described. Shew that if a circle is inscribed in the space enclosed by the three semi-circles its radius must be $\frac{2}{3}$″.

(Theoretical)

6. *A straight line is drawn through the point of contact of two circles whose centres are A and B, cutting the circumferences at P and Q respectively; shew that the radii AP and BQ are parallel.*

7. Two circles touch externally, and through the point of contact a straight line is drawn terminated by the circumferences; shew that the tangents at its extremities are parallel.

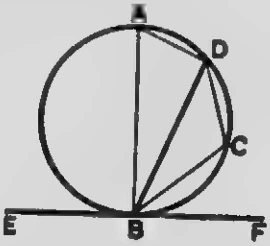
8. Find the locus of the centres of all circles which touch a given circle (i) at a given point; (ii) and are of a given radius.

9. From a given point as centre describe a circle to touch a given circle. How many solutions will there be?

10. Describe a circle of radius a to touch a given circle of radius b at a given point. How many solutions will there be?

THEOREM 45. [Euclid III. 32]

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



Let EF touch the $\odot\, ABC$ at B, and let BD be a chord drawn from B, the point of contact.

It is required to prove that

(i) *the* $\angle\, FBD$ = *the angle in the alternate segment* BAD ;

(ii) *the* $\angle\, EBD$ = *the angle in the alternate segment* BCD.

Let BA be the diameter through B, and C any point in the arc of the segment which does not contain A.

Join AD, DC, CB.

Proof. Because the $\angle\, ADB$ in a semi-circle is a rt. angle,

$\therefore$ the $\angle$s DBA, BAD together = a rt. angle.

But since EBF is a tangent, and BA a diameter,

$\therefore$ the $\angle\, FBA$ is a rt. angle.

$\therefore$ the $\angle\, FBA$ = the $\angle$s DBA, BAD together.

Take away the common $\angle\, DBA$,

then the $\angle\, FBD$ = the $\angle\, BAD$, in the alternate segment.

Again because $ABCD$ is a cyclic quadrilateral,

$\therefore$ the $\angle\, BCD$ = the supplement of the $\angle\, BAD$

= the supplement of the $\angle\, FBD$

= the $\angle\, EBD$;

$\therefore$ the $\angle\, EBD$ = the $\angle\, BCD$, in the alternate segment.

Q.E.D.

EXERCISES ON THEOREM 45

1. In the figure of Theorem 45, if the $\angle FBD = 72°$, write down the values of the $\angle s$ BAD, BCD, EBD.

2. Use this theorem to shew that tangents to a circle from an external point are equal.

3. Through A, the point of contact of two circles, chords APQ, AXY are drawn; shew that PX and QY are parallel.

Prove this (i) for internal, (ii) for external contact.

4. AB is the common chord of two circles, one of which passes through O, the centre of the other; prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

5. Two circles intersect at A and B; and through P, any point on one of them, straight lines PAC, PBD are drawn to cut the other at C and D; shew that CD is parallel to the tangent at P.

6. If from the point of contact of a tangent to a circle a chord is drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

7. Deduce Theorem 44 from the property that *the line of centres bisects a common chord at right angles.*

8. Deduce Theorem 45 from Ex. 5, page 165.

9. Deduce Theorem 42 from Theorem 39.

PROBLEMS

Geometrical Analysis

Hitherto the Propositions of this text-book have been arranged **Synthetically**, that is to say, by *building up known results* in order to obtain a *new* result.

But this arrangement, though convincing as an argument, in most cases affords little clue as to the way in which the construction or proof *was discovered*. We therefore draw the student's attention to the following hints.

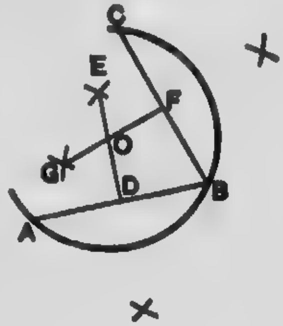
In attempting to solve a problem begin by *assuming* the required result ; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some condition or known theorem which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of the conditions of a proposition in order to trace it back to some earlier principle on which it depends is called **geometrical analysis** : it is the natural way of attacking the harder types of exercises, and it is especially useful in solving problems.

Although the above directions do not amount to a *method*, they often furnish a very effective mode of *searching for a suggestion*. The approach by analysis will be illustrated in some of the following problems. [See Problems 24, 29, 30.]

PROBLEM 21

Given a circle, or an arc of a circle, to find its centre.



Let ABC be an arc of a circle whose centre is to be found.

Construction. Take two chords AB, BC, and bisect them at right angles by the lines DE, FG, meeting at O. *Prob.* 2.

Then O is the required centre.

Proof. Every point in DE is equidistant from A and B. *Prob.* 14.

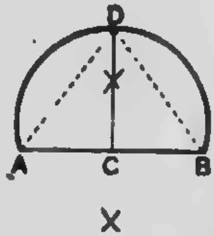
And every point in FG is equidistant from B and C.

$\therefore$ O is equidistant from A, B, and C.

$\therefore$ O is the centre of the circle ABC. *Theor.* 36.

PROBLEM 22

To bisect a given arc.



Let ADB be the given arc to be bisected.

Construction. Join AB, and bisect it at right angles by CD meeting the arc at D. *Prob.* 2.

Then the arc is bisected at D.

Proof. Join DA, DB.

Then every point on CD is equidistant from A and B; *Prob.* 14.

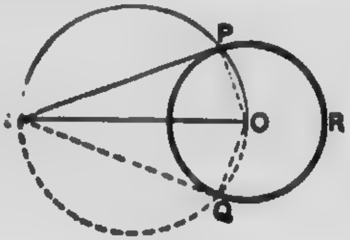
$\therefore$ $DA = DB$;

$\therefore$ the $\angle DBA =$ the $\angle DAB$; *Theor.* 6.

$\therefore$ the arcs, which subtend these angles at the $\bigcirc^{ce}$, are equal; that is, the arc $DA =$ the arc DB.

PROBLEM 23

To draw a tangent to a circle from a given external point.



Let PQR be the given circle with its centre at O; and let T be the point from which a tangent is to be drawn.

Construction. Join TO, and on it describe a semi-circle TPO to cut the circle at P.

Join TP.

Then TP is the required tangent.

Proof. Join OP.

Then since the $\angle\ TPO$, being in a semi-circle, is a rt. angle,

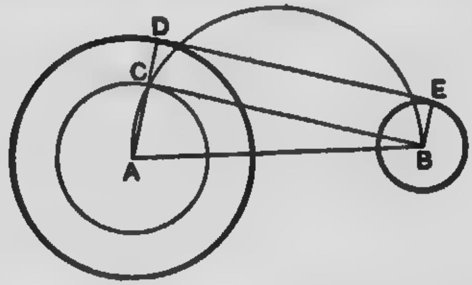
$\therefore$ TP is at right angles to the radius OP.

$\therefore$ TP is a tangent at P. *Theor.* 42.

Since the semi-circle may be described on either side of TO, a second tangent TQ can be drawn from T, as shewn in the figure.

Problem 24

To draw a common tangent to two circles.



Let A be the centre of the greater circle, and a its radius; and let B be the centre of the smaller circle, and b its radius.

Analysis. Suppose DE to touch the circles at D and E. Then the radii AD, BE being perp. to DE, are parallel.

Now if BC were drawn par[l] to DE, then the fig. DB would be a rectangle, so that $CD = BE = b$.

And if AD, BE are *on the same side of* AB,

then $AC = a - b$, and the $\angle ACB$ is a rt. angle.

These hints enable us to draw BC *first*, and thus lead to the following construction.

Construction. With centre A, and radius equal to the *difference* of the radii of the given circles, describe a circle and draw BC to touch it.

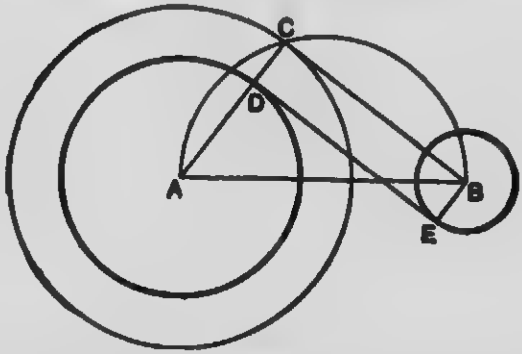
Join AC, and produce it to meet the circle (A) at D.

Through B draw the radius BE par[l] to AD and in the *same sense*. Join DE.

Then DE is a common tangent to the given circles.

Obs. Since two tangents, such as BC, can in general be drawn from B to the circle of construction, this method will furnish two common tangents to the given circles. These are called the **direct common tangents.**

PROBLEM 24. (*Continued*)



Again, if the circles are *external to one another* two more common tangents may be drawn.

Analysis. In this case we may suppose DE to touch the circles at D and E so that the radii AD, BE fall *on opposite sides of* AB.

Then BC, drawn par[l] to the supposed common tangent DE, would meet AD *produced* at C; and we should now have

$AC = AD + DC = a + b$; and the $\angle ACB$ is a rt. angle.

Hence the following construction.

Construction. With centre A, and radius equal to the *sum* of the radii of the given circles, describe a circle, and draw BC to touch it.

Then proceed as in the first case, but draw BE in the sense *opposite* to AD.

Obs. As before, two tangents may be drawn from B to the circle of construction; hence two common tangents may be thus drawn to the given circles. These are called the **transverse common tangents.**

[We leave as an exercise to the student the arrangement of the proof in synthetic form.]

EXERCISES ON COMMON TANGENTS

(*Numerical and Graphical*)

1. How many common tangents can be drawn (i) when the given circles intersect; (ii) when they have external contact; (iii) when they have internal contact?

Illustrate your answer by drawing two circles of radii 1·4″ and 1·0″ respectively, (i) with 1·0″ between the centres; (ii) with 2·4″ between the centres; (iii) with 0·4″ between the centres; (iv) with 3·0″ between the centres.

Draw the common tangents in each case, and note where the general construction fails, or is modified.

2. Draw two circles with radii 2·0″ and 0·8″, placing their centres 2·0″ apart. Draw the common tangents, and find their lengths between the points of contact, both by calculation and by measurement.

3. Draw all the common tangents to two circles whose centres are 1·8″ apart and whose radii are 0·6″ and 1·2″ respectively. Calculate and measure the length of the direct common tangents.

4. Two circles of radii 1·7″ and 1·0″ have their centres 2·1″ apart. Draw their common tangents and find their lengths. Also find the length of the common chord. Produce the common chord and shew by measurement that it bisects the common tangents.

5. Draw two circles with radii 1·6″ and 0·8″ and with their centres 3·0″ apart. Draw all their common tangents.

6. Draw the direct common tangents to two equal circles.

(*Theoretical*)

7. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

8. If four common tangents are drawn to two circles external to one another, shew that the two direct, and also the two transverse, tangents intersect on the line of centres.

9. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q; shew that PQ subtends a right angle at the point A.

On the Construction of Circles

In order to draw a circle we must know (i) the position of the centre, (ii) the length of the radius.

(i) To find the position of the centre, two conditions are needed, each giving a locus on which the centre must lie ; so that the one or more points in which the two loci intersect are possible positions of the required centre, as explained on page 93.

(ii) The position of the centre being thus fixed, the radius is determined if we know (or can find) any point on the circumference.

Hence to draw a circle *three* independent data are required.

For example, we may draw a circle if we are given (i) *three* points on the circumference ; *or* (ii) *three* tangent lines ; *or* (iii) one point on the circumference, one tangent, and its point of contact.

It will however often happen that more than one circle can be drawn satisfying three given conditions.

Before attempting the constructions of the next Exercise the student should make himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

EXERCISES

1. Draw a circle to pass through three given points.

2. If a circle touches a given line PQ at a point A, on what line must its centre lie?

If a circle passes through two given points A and B, on what line must its centre lie?

Hence draw a circle to touch a straight line PQ at the point A, and to pass through another given point B.

3. If a circle touches a given circle whose centre is C at the point A, on what line must its centre lie?

Draw a circle to touch the given circle (C) at the point A, and to pass through a given point B.

4. A point P is 4·5 cm. distant from a straight line AB. Draw two circles of radius 3·2 cm. to pass through P and to touch AB.

5. Given two circles of radius 3·0 cm. and 2·0 cm. respectively, their centres being 6·0 cm. apart; draw a circle of radius 3·5 cm. to touch each of the given circles externally.

How many solutions will there be? What is the radius of the smallest circle that touches each of the given circles externally?

6. If a circle touches two straight lines AO, OB, on what line must its centre lie?

Draw OA, OB, making an angle of 76°, and describe a circle of radius 1·2″ to touch both lines.

7. Given a circle of radius 3·5 cm., with its centre 5·0 cm. from a given straight line AB; draw two circles of radius 2·5 cm. to touch the given circle and the line AB.

8. Devise a construction for drawing a circle to touch each of two parallel straight lines and a transversal.

Shew that two such circles can be drawn, and that they are equal.

9. Describe a circle to touch a given circle, and also to touch a given straight line at a given point.

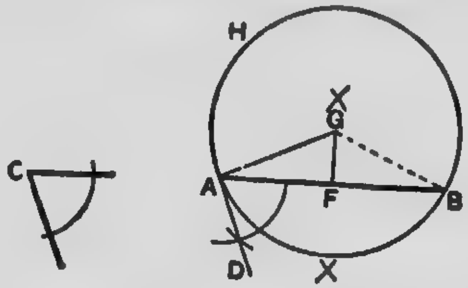
10. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

11. Shew how to draw a circle to touch each of three given straight lines of which no two are parallel.

How many such circles can be drawn?

PROBLEM 25

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st. line, and C the given angle.

It is required to describe on AB a segment of a circle containing an angle equal to C.

Construction. At A in BA, make the $\angle BAD$ equal to the $\angle C$.

From A draw AG perp. to AD.

Bisect AB at rt. angles by FG, meeting AG in G. *Prob.* 2.

Proof. Join GB.

Now every point in FG is equidistant from A and B;
Prob. 14.

$$\therefore GA = GB.$$

With centre G, and radius GA, draw a circle, which must pass through B, and touch AD at A.

Then the segment AHB, alternate to the $\angle BAD$, contains an angle equal to C.
Theor. 45.

NOTE. In the particular case when the given angle is a rt. angle, the segment required will be the semi-circle on AB as diameter. [Theorem 39.]

Corollary. *To cut off from a given circle a segment containing a given angle, it is enough to draw a tangent to the circle, and from the point of contact to draw a chord making with the tangent an angle equal to the given angle.*

It was proved on page 163 that
The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

The following Problems are derived from this result by the Method of Intersection of Loci [page 93].

EXERCISES

1. *Describe a triangle on a given base having a given vertical angle and having its vertex on a given straight line.*

2. *Construct a triangle having given the base, the vertical angle and*
   (i) *one other side.*
   (ii) *the altitude.*
   (iii) *the length of the median which bisects the base.*
   (iv) *the foot of the perpendicular from the vertex to the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the $\bigcirc^{ce}$ by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the $\bigcirc^{ce}$ at C. Then ABC is the required triangle.]

4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H a line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the $\angle K$. With centre A, and radius H, describe a circle cutting the arc of the latter segment at X and Y. Join AX (or AY) cutting the arc of the first segment at C. Then ABC is the required triangle.]

5. *Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.*

CIRCLES IN RELATION TO RECTILINEAL FIGURES

Definitions

1. A **Polygon** is a rectilineal figure bounded by more than four sides.

A Polygon of *five* sides is called a **Pentagon,**
A Polygon of *six* sides is called a **Hexagon,**
A Polygon of *seven* sides is called a **Heptagon,**
A Polygon of *eight* sides is called an **Octagon,**
A Polygon of *ten* sides is called a **Decagon,**
A Polygon of *twelve* sides is called a **Dodecagon,**
A Polygon of *fifteen* sides is called a **Quindecagon.**

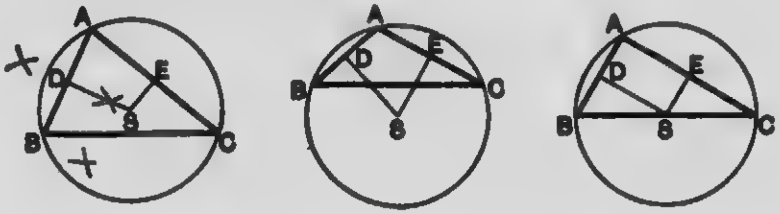
2. A Polygon is **Regular** when all its sides are equal, and all its angles are equal.

3. A rectilineal figure is said to be **inscribed** in a circle, when all its angular points are on the circumference of the circle; and a circle is said to be **circumscribed about** a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure.

4. A circle is said to be **inscribed in** a rectilineal figure, when the circumference of the circle is touched by each side of the figure; and a rectilineal figure is said to be **circumscribed about** a circle, when each side of the figure is a tangent to the circle.

PROBLEM 26

To circumscribe a circle about a given triangle.



Let ABC be the triangle, about which a circle is to be drawn.

Construction. Bisect AB and AC at rt. angles by DS and ES, meeting at S. *Prob.* 2.

Then S is the centre of the required circle.

Proof. Now every point in DS is equidistant from A and B; *Prob.* 14.

and every point in ES is equidistant from A and C;

$\therefore$ S is equidistant from A, B, and C.

With centre S, and radius SA describe a circle; this will pass through B and C, and is, therefore, the required circum-circle.

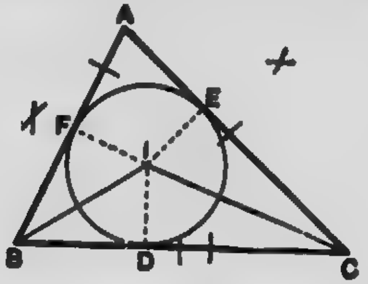
Obs. It will be found that if the given triangle is acute-angled, the centre of the circum-circle falls within it: if it is a right-angled triangle, the centre falls on the hypotenuse: if it is an obtuse-angled triangle, the centre falls without the triangle.

NOTE. From page 91 it is seen that if S is joined to the middle point of BC, then the joining line is perpendicular to BC.

Hence the perpendiculars drawn to the sides of a triangle from their middle points are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.

PROBLEM 27

To inscribe a circle in a given triangle.



Let ABC be the triangle, in which a circle is to be inscribed.

Construction. Bisect the $\angle s$ ABC, ACB by the st. lines BI, CI, which intersect at I. *Prob.* 1.

Then I is the centre of the required circle.

Proof. From I draw ID, IE, IF perp. to BC, CA, AB.

Then every point in BI is equidistant from BC, BA; *Prob.* 15.

$$\therefore ID = IF.$$

And every point in CI is equidistant from CB, CA;

$$\therefore ID = IE.$$

$\therefore ID, IE, IF$ are all equal.

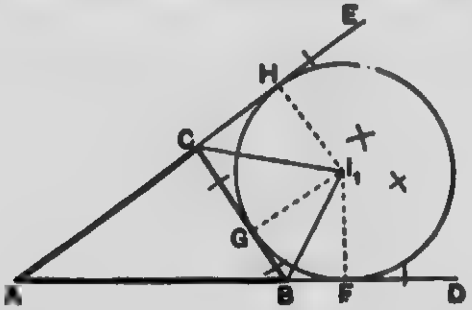
With centre I and radius ID draw a circle;
this will pass through the points E and F.
Also the circle will touch the sides BC, CA, AB,
because the angles at D, E, F are right angles.
$\therefore$ the $\odot$ DEF is inscribed in the $\triangle ABC$.

NOTE. From II., p. 97 and Problem 27 it follows that
The bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the inscribed circle.

Definition. A circle which touches one side of a triangle and the other two sides produced is called an **escribed** circle of the triangle.

Problem 28

To draw an escribed circle of a given triangle.



Let ABC be the given triangle of which the sides AB, AC are produced to D and E.

It is required to describe a circle touching BC, and BD, CE.

Construction. Bisect the $\angle$ CBD, BCE by the st. lines BI_1, CI_1 which intersect at I_1.

Then I_1 is the centre of the required circle.

Proof. From I_1 draw I_1F, I_1G, I_1H perp. to AD, BC, AE. Then every point in BI_1 is equidistant from BD, BC; *Prob.* 15.

$\therefore I_1F = I_1G.$

Similarly $I_1G = I_1H.$

$\therefore I_1F, I_1G, I_1H$ are all equal.

With centre I_1 and radius I_1F describe a circle ;
this will pass through the points G and H.

Also the circle will touch AD, BC, and AE,
because the angles at F, G, H are rt. angles.

$\therefore$ the $\odot$ FGH is an escribed circle of the $\triangle$ ABC.

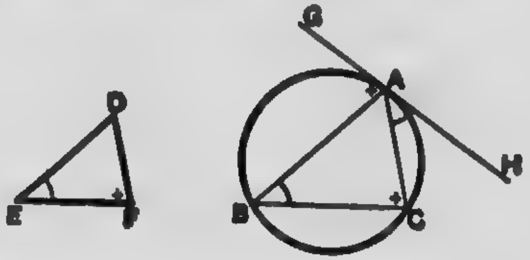
Note 1. It is clear that every triangle has three escribed circles. Their centres are known as the **Ex-centres.**

Note 2. From II *a*, page 97 and Problem 28 it follows that

The bisectors of two exterior angles of a triangle and of the third angle are concurrent, the point of intersection being an ex-centre.

PROBLEM 29

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. A triangle ABC, equiangular to the $\triangle DEF$, is inscribed in the circle, if from any point A on the $\bigcirc^{ce}$ two chords AB, AC can be so placed that, on joining BC, the $\angle B =$ the $\angle E$, and the $\angle C =$ the $\angle F$; for then the $\angle A =$ the $\angle B =$ the $\angle D$. *Theor.* 16.

Now the $\angle B$, in the segment ABC, suggests the *equal* angle between the chord AC and the tangent at its extremity (*Theor.* 49); so that, if at A we draw the tangent GAH,

then the $\angle HAC =$ the $\angle E$;

and similarly, the $\angle GAB =$ the $\angle F$.

Reversing these steps, we have the following construction.

Construction. At any point A on the $\bigcirc^{ce}$ of the $\odot ABC$ draw the tangent GAH. *Prob.* 23.

At A make the $\angle GAB$ equal to the $\angle F$,

and make the $\angle HAC$ equal to the $\angle E$.

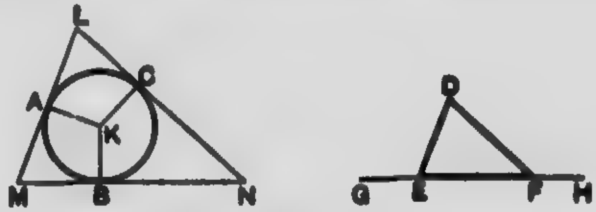
Join BC.

Then ABC is the required triangle.

NOTE. In drawing the figure on a larger scale the student should shew the construction lines for the tangent GAH and for the angles GAB, HAC. A similar remark applies to the next Problem.

PROBLEM 30

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. Suppose LMN to be a circumscribed triangle in which the $\angle M =$ the $\angle E$, the $\angle N =$ the $\angle F$, and consequently, the $\angle L = \angle D$.

Let us consider the radii KA, KB, KC, drawn to the points of contact of the sides; for the tangents LM, MN, NL could be drawn if we knew the relative positions of KA, KB, KC, that is, if we knew the $\angle s$ BKA, BKC.

Now from the quad^l $BKAM$, since the $\angle s$ B and A are rt. $\angle s$,

the $\angle BKA = 180° - M = 180° - E$;

similarly the $\angle BKC = 180° - N = 180° - F$.

Hence we have the following construction.

Construction. Produce EF both ways to G and H.

Find K the centre of the $\odot ABC$,

and draw any radius KB.

At K make the $\angle BKA$ equal to the $\angle DEG$;

and make the $\angle BKC$ equal to the $\angle DFH$.

Through A, B, C draw LM, MN, NL perp. to KA, KB, KC.

Then LMN is the required triangle.

[The student should now arrange the proof synthetically.]

EXERCISES

On Circles and Triangles

(Inscriptions and Circumscriptions)

1. In a circle of radius 5 cm. inscribe an equilateral triangle; and about the same circle circumscribe a second equilateral triangle. In each case state and justify your construction.

2. Draw an equilateral triangle on a side of 8 cm., and find by calculation and measurement (to the nearest millimetre) the radii of the inscribed, circumscribed, and escribed circles.

Why are the latter radii double and treble of the first?

3. Draw triangles from the following data:

(i) $a = 2{\cdot}5''$, $B = 66°$, $C = 50°$;
(ii) $a = 2{\cdot}5''$, $B = 72°$, $C = 44°$;
(iii) $a = 2{\cdot}5''$, $B = 41°$, $C = 23°$.

Circumscribe a circle about each triangle, and measure the radii to the nearest hundredth of an inch. Account for the results being the same, by comparing the vertical angles.

4. In a circle of radius 4 cm. inscribe an equilateral triangle. Calculate and measure its side to the nearest millimetre.

Find the area of the inscribed equilateral triangle, and shew that it is one quarter of the circumscribed equilateral triangle.

5. In the triangle ABC, if I is the centre, and r the length of the radius of the in-circle, shew that

$$\triangle IBC = \tfrac{1}{2}ar; \quad \triangle ICA = \tfrac{1}{2}br; \quad \triangle IAB = \tfrac{1}{2}cr.$$

Hence prove that $\triangle ABC = \tfrac{1}{2}(a + b + c)r.$

6. If r_1 is the radius of the ex-circle opposite to A, prove that

$$\triangle ABC = \tfrac{1}{2}(b + c - a)r_1.$$

If $a = 5$ cm., $b = 4$ cm., $c = 3$ cm., verify by measurement the results of Ex. 5 and of this exercise.

7. Find by measurement the circum-radius of the triangle ABC in which $a = 6{\cdot}3$ cm., $b = 3{\cdot}0$ cm., and $c = 5{\cdot}1$ cm.

Draw and measure the perpendiculars from A, B, C to the opposite sides. If their lengths are represented by p_1, p_2, p_3, verify the following statement:

$$\text{circum-radius} = \frac{bc}{2p_1} = \frac{ca}{2p_2} = \frac{ab}{2p_3}.$$

O

EXERCISES

On Circles and Squares

(Inscriptions and Circumscriptions)

1. Draw a circle of radius 1·5″, and find a construction for inscribing a square in it.

Calculate the length of the side to the nearest hundredth of an inch, and verify by measurement.

Find the area of the inscribed square.

2. Circumscribe a square about a circle of radius 1·5″, shewing all lines of construction.

Prove that the area of the square circumscribed about a circle is double that of the inscribed square.

3. Draw a square on a side of 7·5 cm., and state a construction for inscribing a circle in it.

Justify your construction by considerations of symmetry.

4. Circumscribe a circle about a square whose side is 6 cm.

Measure the diameter to the nearest millimetre, and test your drawing by calculation.

5. In a circle of radius 1·8″ inscribe a rectangle of which one side measures 3·0″. Find the approximate length of the other side.

Of all rectangles inscribed in the circle shew that the square has the greatest area.

6. A square and an equilateral triangle are inscribed in a circle. If a and b denote the lengths of their sides, shew that $3a^2 = 2b^2$.

7. $ABCD$ is a square inscribed in a circle, and P is any point on the arc AD: shew that the side AD subtends at P an angle three times as great as that subtended at P by any one of the other sides.

(Problems. State your construction, and give a theoretical proof.)

8. Circumscribe a rhombus about a given circle.

9. Inscribe a square in a given square $ABCD$, so that one of its angular points shall be at a given point X in AB.

10. In a given square inscribe the square of minimum area.

11. Describe (i) a circle, (ii) a square about a given rectangle.

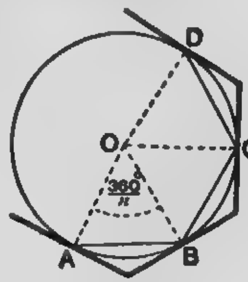
12. Inscribe (i) a circle, (ii) a square in a given quadrant.

ON CIRCLES AND REGULAR POLYGONS

PROBLEM 31

To draw a regular polygon (i) **in** (ii) **about** *a given circle.*

Let AB, BC, CD, $\cdots$ be consecutive sides of a regular polygon inscribed in a circle whose centre is O.



Then AOB, BOC, COD, $\cdots$ are congruent isosceles triangles. And if the polygon has n sides, each of the

$$\angle s\ AOB,\ BOC,\ COD,\ \cdots = \frac{360^\circ}{n}.$$

(i) Thus to **inscribe** a polygon of n sides in a given circle, draw at the centre an angle AOB of this size. This gives the length of a side AB; and chords equal to AB may now be set off round the circumference. The resulting figure will clearly be equilateral and equiangular.

(ii) To **circumscribe** a polygon of n sides about the circle, the points A, B, C, D, $\cdots$ must be determined as before, and tangents drawn to the circle at these points. The resulting figure may readily be proved equilateral and equiangular.

NOTE. This method gives a *strict geometrical construction* only when the angle AOB can be drawn with ruler and compasses.

EXERCISES

1. By strict constructions inscribe in a circle (radius 4 cm.) a regular (i) hexagon; (ii) octagon; (iii) dodecagon.

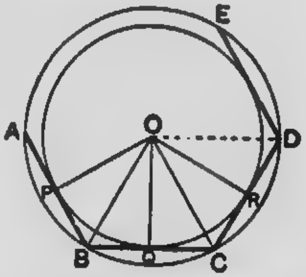
2. About a circle of radius $1 \cdot 5''$ circumscribe a regular (i) hexagon; (ii) octagon. Test the constructions by measurement, and justify them by proof.

3. Compare the sides and also the areas of an equilateral triangle and a regular hexagon inscribed in any circle.

4. Using a protractor inscribe a regular heptagon in a circle of radius $2''$. Calculate and measure one angle; measure a side.

Problem 32

To draw a circle (i) **in** (ii) **about** *a regular polygon.*



Let AB, BC, CD, DE, $\cdots$ be consecutive sides of a regular polygon of n sides.

Bisect the $\angle s$ ABC, BCD by BO, CO meeting at O.

Then O is the centre both of the inscribed and circumscribed circle.

Outline of Proof. Join OD; and from the congruent $\triangle s$ OCB, OCD, shew that OD bisects the $\angle CDE$ and that:

All the bisectors of the angles of the polygon meet at O.

(i) Prove that $OB = OC = OD = \cdots$; from Theorem 6.

Hence O is the circum-centre.

(ii) Draw OP, OQ, OR, $\cdots$ perp. to AB, BC, CD, $\cdots$.

Prove that $OP = OQ = OR = \cdots$; from the congruent $\triangle s$ OBP, OBQ, $\cdots$.

Hence O is the in-centre.

EXERCISES

1. Draw a regular hexagon on a side of 2·0″. Draw the inscribed and circumscribed circles. Calculate and measure their diameters to the nearest hundredth of an inch.

2. Shew that the area of a regular hexagon inscribed in a circle is three-fourths of that of the circumscribed hexagon.

   Find these to the nearest tenth of a sq. cm. (radius 10 cm.).

3. If ABC is an isosceles triangle inscribed in a circle, having each of the angles B and C double of the angle A; shew that BC is a side of a regular pentagon inscribed in the circle.

4. On a side of 4 cm. construct (without protractor) a regular (i) hexagon; (ii) octagon; and in each case find the approximate area of the figure.

THE CIRCUMFERENCE OF A CIRCLE

By experiment and measurement it is found that the length of the circumference of a circle is roughly $3\frac{1}{7}$ times the length of its diameter: that is to say

$$\frac{\textit{circumference}}{\textit{diameter}} = 3\tfrac{1}{7} \text{ nearly;}$$

and it can be proved that this is the same for all circles.

A more correct value of this ratio is found by theory to be 3·1416 ; while correct to 7 places of decimals it is 3·1415926. Thus the value $3\frac{1}{7}$ (or 3·14*28*) is correct to 2 places only.

The ratio which the circumference of any circle bears to its diameter is denoted by the Greek letter π; so that

$$\textit{circumference} = \textit{diameter} \times \pi = 2\,r \times \pi = 2\,\pi r;$$

where r denotes the radius of the circle and where to π we are to give one of the values $3\frac{1}{7}$, 3·1416, or 3·1415926, according to the degree of accuracy required in the final result.

NOTE. The theoretical methods by which π is evaluated to any required degree of accuracy cannot be explained at this stage, but its value may be easily verified by experiment to two decimal places.

For example: round a cylinder wrap a strip of paper so that the ends overlap. At any point in the overlapping area prick a pin through both folds. Unwarp and straighten the strip, then measure the distance between the pin holes: this gives the circumference. Measure the diameter, and divide the first result by the second.

Ex. 1. From these data find and record the value of π.

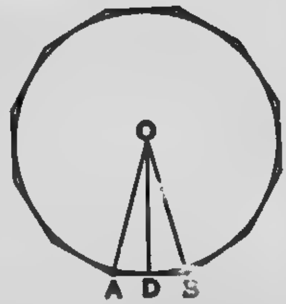
Find the mean of the three results.

CIRCUMFERENCE.	DIAMETER.	VALUE OF π.
16·0cm.	5·1 cm.	
8·8″	2·8″	
13·5′	4·3″	

Ex. 2. A fine thread is wound evenly round a cylinder, and it is found that the length required for 20 complete turns is 75·4″. The diameter of the cylinder is 1·2″: find roughly the value of π.

Ex. 3. A bicycle wheel, 28″ in diameter, makes 400 revolutions in travelling over 977 yards. Hence estimate the value of π.

THE AREA OF A CIRCLE



Let AB be a side of a polygon of n sides circumscribed about a circle whose centre is O and radius r. Then

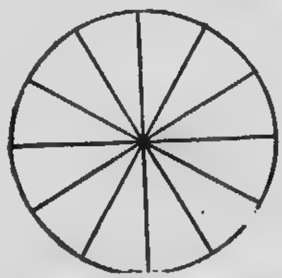
$$\textit{Area of polygon} = n \cdot \triangle AOB$$
$$= n \cdot \tfrac{1}{2} AB \times OD = \tfrac{1}{2} \cdot nAB \times r$$
$$= \tfrac{1}{2} (\textit{perimeter of polygon}) \times r\,;$$

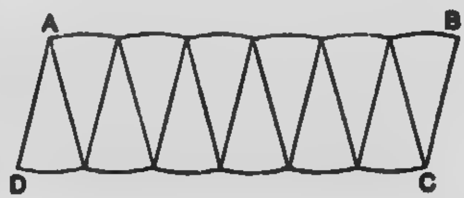
and this is true however many sides the polygon may have.

Now if the number of sides is increased without limit, the perimeter and area of the polygon may be made to differ from the circumference and area of the circle by quantities smaller than any that can be named; hence ultimately

$$\textit{Area of circle} = \tfrac{1}{2} \cdot \textit{circumference} \times r = \tfrac{1}{2} \cdot 2\pi r \times r = \pi r^2.$$

ALTERNATIVE METHOD





Suppose the circle divided into any even number of sectors having equal central angles: denote the number of sectors by n.

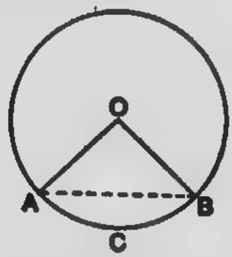
Let the sectors be placed side by side as in the diagram; then the area of the circle = the area of the fig. $ABCD$.

Now if the number of sectors be increased, each arc is decreased; so that (i) the outlines AB, CD tend to become *straight*, and (ii) the angles at D and B tend to become rt. angles.

Thus when n is increased without limit, the fig. $ABCD$ ultimately becomes a *rectangle*, whose length is the *semi-circumference of the circle*, and whose breadth is its *radius*.

$$\therefore\ \textit{Area of circle} = \tfrac{1}{2} \cdot \textit{circumference} \times \textit{radius} = \tfrac{1}{2} \cdot 2\pi r \times r = \pi r^2.$$

THE AREA OF A SECTOR



If two radii of a circle make an angle of 1°, they cut off

(i) an arc whose length = $\frac{1}{360}$ of the circumference ;

and (ii) a sector whose area = $\frac{1}{360}$ of the circle ;

$\therefore$ if the angle AOB contains D degrees, then

(i) *the arc* AB $= \dfrac{D}{360}$ *of the circumference;*

(ii) *the sector* $AOB = \dfrac{D}{360}$ *of the area of the circle*

$$= \frac{D}{360} \text{ of } (\tfrac{1}{2} \text{ circumference} \times \text{radius})$$

$$= \tfrac{1}{2} \cdot \text{arc } AB \times \text{radius}.$$

THE AREA OF A SEGMENT

The area of a minor segment is found by subtracting from the corresponding sector the area of the triangle formed by the chord and the radii. Thus

Area of segment ABC = *sector* $OACB$ − *triangle* AOB.

The area of a major segment is most simply found by subtracting the area of the corresponding minor segment from the area of the circle.

EXERCISES

[*In each case choose the value of π so as to give a result of the assigned degree of accuracy.*]

1. Find to the nearest millimetre the circumferences of the circles whose radii are (i) 4·5 cm. (ii) 100 cm.

2. Find to the nearest hundredth of a square inch the areas of the circles whose radii are (i) 2·3″. (ii) 10·6″.

3. Find to two places of decimals the circumference and area of a circle inscribed in a square whose side is 3·6 cm.

4. In a circle of radius 7·0 cm. a square is described: find to the nearest square centimetre the difference between the areas of the circle and the square.

5. Find to the nearest hundredth of a square inch the area of the circular ring formed by two concentric circles whose radii are 5·7″ and 4·3″.

6. Shew that the area of a ring lying between the circumferences of two concentric circles is equal to the area of a circle whose radius is the length of a tangent to the inner circle from any point on the outer.

7. A rectangle whose sides are 8·0 cm and 6·0 cm. is inscribed in a circle. Calculate to the nearest tenth of a square centimetre the total area of the four segments outside the rectangle.

8. Find to the nearest tenth of an inch the side of a square whose area is equal to that of a circle of radius 5″.

9. A circular ring is formed by the circumference of two concentric circles. The area of the ring is 22 square inches, and its width is 1·0″; taking π as $\frac{22}{7}$, find approximately the radii of the two circles.

10. Find to the nearest hundredth of a square inch the difference between the areas of the circumscribed and inscribed circles of an equilateral triangle each of whose sides is 4″.

EXERCISES

(*Theoretical*)

1. Describe a circle to touch two parallel straight lines and a third straight line which meets them. Shew that two such circles can be drawn, and that they are equal.

2. *Triangles which have equal bases and equal vertical angles have equal circumscribed circles.*

3. If, in a triangle, ABC, I, S, the centres of the inscribed and circumscribed circles, and A are collinear, then $AB = AC$.

4. The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

5. If the circle inscribed in the triangle ABC touches the sides at D, E, F; shew that the angles of the triangle DEF are respectively the complements of the halves of the angles A, B, C.

6. If I is the centre of the inscribed circle and I_1 the centre of the escribed circle of the triangle ABC, then I, B, I_1, C are concyclic.

7. In any triangle the difference of two sides is equal to the difference of the segments into which the third side is divided at the point of contact of the inscribed circle.

8. In the triangle ABC, I and S are the centres of the inscribed and circumscribed circles: then IS subtends at A an angle equal to half the difference of the angles at the base of the triangle. Also if AD is perpendicular to BC, AI bisects the $\angle DAS$.

9. The diagonals of a quadrilateral $ABCD$ intersect at O: shew that the centres of the circles circumscribed about the four triangles AOB, BOC, COD, DOA are the vertices of a parallelogram.

10. In any triangle ABC, if I is the centre of the inscribed circle, and if AI is produced to meet the circumscribed circle at O, O is the centre of the circum-circle of the triangle BIC.

11. Given the base, altitude, and the radius of the circumscribed circle; construct the triangle.

12. Three circles whose centres are A, B, C touch one another externally two by two at D, E, F: shew that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

EXERCISES

(Loci)

1. Given the base BC and the vertical angle A of a triangle; find the locus of the ex-centre opposite A.

2. Find the locus of the intersection of the bisectors of the angles PAB, QBA if A, B are fixed and PA, BQ are constantly parallel.

3. Find the locus of the middle points of chords of a circle which pass through a fixed point (i) within, (ii) on, (iii) without the circumference.

4. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

5. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

6. A and B are two fixed points on the circumference of a circle, and PQ is any diameter; if PA, QB cut in X, find the locus of X.

7. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that AP is equal to AC; find the locus of P.

8. AB is a fixed chord of a circle, and AC is a movable chord passing through A; if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

9. A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers; find the locus of X.

10. Two circles intersect at A and B, and P is any point on the circumference of one of them. If the lines PA, PB cut the other circle at X and Y, find the locus of the intersection of AY and BX.

11. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn; find the locus of the intersection of HP and QK.

PART IV

ON PROPORTION

Definitions and First Principles

1. The **ratio** of one magnitude to another *of the same kind* is the relation which the first bears to the second in regard to quantity; this is measured by the fraction which the first is of the second.

Thus if two such magnitudes contain a and b units respectively the ratio of the first to the second is expressed by the fraction $\frac{a}{b}$.

The ratio of a to b is generally denoted thus, $a : b$; and a is called the **antecedent** and b the **consequent** of the ratio.

The two magnitudes compared in a ratio must be *of the same kind;* for example, both must be lines, or both angles, or both areas. It is clearly impossible to compare the *length* of a straight line with a magnitude of a different kind, such as the *area* of a triangle. Moreover, a ratio is an *abstract* fraction. Thus the ratio of a line 6 cm. long to a line 8 cm. long is $\frac{6}{8}$ or $\frac{3}{4}$ (not $\frac{3}{4}$ *cm.*).

Note. It is not always possible to express two quantities of the same kind in terms of a common unit. For instance, if the side of a square is 1 inch, the diagonal is $\sqrt{2}$ inches. But since the numerical value of $\sqrt{2}$ cannot be *exactly* determined (though it can be found to any number of decimal figures), the side and diagonal cannot be expressed in terms of the same unit. Two such quantities are said to be **incommensurable**. But by choosing a sufficiently small quantity as unit, two incommensurables, such as $\sqrt{2}$ inches and 1 inch, may be expressed to any required degree of accuracy. Thus, remembering that $\sqrt{2} = 1{\cdot}41421\cdots$, it follows that $\sqrt{2}$ inches and 1 inch may be represented by

1414 and 1000, roughly, taking $\frac{1}{1000}''$ as unit;
14142 and 10000, more nearly, taking $\frac{1}{10000}''$ as unit; and so on.

2. If a point X is taken in a straight line AB, or in AB produced, then X is said to divide AB into the two **segments** AX, XB; the segments being in either case *the distances of the dividing point X from the extremities of the given line AB.*

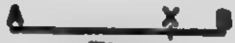


Fig. 1.

Fig. 2.

3. X is said to divide AB internally in Fig. 1, and externally in Fig. 2. In the first case AB is the sum, and in the second the difference, of the segments AX, XB. In either case the ratio in which X divides AB is the ratio of the segments AX, XB.

4. Four magnitudes a, b, x, y are **proportionals** or **in proportion**, when the ratio of the *first* to the *second* is equal to the ratio of the *third* to the *fourth*.

This is expressed by saying "*a is to b as x is to y*"; and the proportion is written

$$\frac{a}{b} = \frac{x}{y},$$

or

$$a : b = x : y.$$

Here a and y are called the **extremes**, and b and x the **means**; and y is said to be a **fourth proportional** to a, b, and x.

In a proportion, terms which are both antecedents or both consequents of the ratios are said to be **corresponding** terms.

Note. In a proportion such as $a : b = x : y$, the magnitudes compared in *each ratio* must be of the same kind, though the magnitudes of the second ratio need not be of the same kind as those of the first. For instance, a and b may denote *areas*, and x and y *lines;* in which case the proportion asserts that the ratio of the areas is the same as the ratio of the lines.

5. Three magnitudes of the same kind are said to be proportionals, when the ratio of the *first* to the *second* is equal to that of the *second* to the *third*.

Thus a, b, c are proportionals if

$$a : b = b : c.$$

Here b is called a **mean proportional** between a and c;
and c is called a **third proportional** to a and b.

Introductory Theorems

I. *If four magnitudes are proportionals, they are also proportionals when taken* **inversely**.

That is, if $a : b = x : y$,
then $b : a = y : x$.

For, by hypothesis, $\dfrac{a}{b} = \dfrac{x}{y}$; hence $\dfrac{b}{a} = \dfrac{y}{x}$;
or $b : a = y : x$.

II. *If four magnitudes of the same kind are proportionals, they are also proportionals when taken* **alternately**.

That is, if $a : b = x : y$,
then $a : x = b : y$.

For, by hypothesis, $\dfrac{a}{b} = \dfrac{x}{y}$;

multiplying both sides by $\dfrac{b}{x}$,

we have $\dfrac{a}{b} \cdot \dfrac{b}{x} = \dfrac{x}{y} \cdot \dfrac{b}{x}$;

that is, $\dfrac{a}{x} = \dfrac{b}{y}$,

or $a : x = b : y$.

Note. In this theorem the *hypothesis* and *conclusion* taken together require that a, b, x and y shall be of the same kind.

III. *If four numbers are proportional, the* **product** *of the* **extremes** *is equal to the* **product** *of the* **means.**

That is, if $a : b = c : d,$

then $ad = bc.$

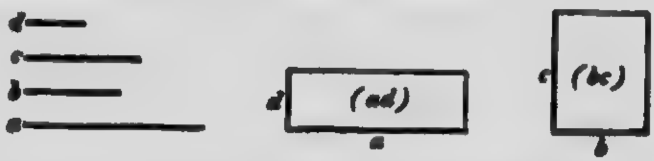
For, by hypothesis, $\frac{a}{b} = \frac{c}{d};$

multiplying each side of this equation by bd, we have

$$ad = bc.$$

Corollary. If a, b, c, d denote the lengths of four straight lines in proportion, *the* **rectangle** *contained by the extremes is equal to the* **rectangle** *contained by the means.*

This is illustrated by the following diagram:



Similarly if three lines a, b, c are proportionals,

that is, if $a : b = b : c;$

then $ac = b^2.$

Or, *the* **rectangle** *contained by the extremes is equal in area to the* **square** *on the mean.*

IV. *If there are four magnitudes in proportion, the* **sum** *(or* **difference**) *of the first and second is to the second as the* **sum** *(or* **difference**) *of the third and fourth is to the fourth.*

That is, if $a : b = x : y;$

then (i) $a + b : b = x + y : y;$

(ii) $a - b : b = x - y : y.$

For by hypothesis, $$\frac{a}{b} = \frac{x}{y};$$

$$\therefore \frac{a}{b} + 1 = \frac{x}{y} + 1, \text{ or } \frac{a+b}{b} = \frac{x+y}{y};$$

that is, $$a + b : b = x + y : y. \qquad \text{(i)}$$

This inference is sometimes referred to as **componendo.**

Similarly by subtracting 1 from the equal ratios $\frac{a}{b}$, $\frac{x}{y}$, we obtain

$$\frac{a-b}{b} = \frac{x-y}{y};$$

that is, $$a - b : b = x - y : y. \qquad \text{(ii)}$$

This inference is sometimes referred to as **dividendo.**

COROLLARY. If $$a : b = x : y,$$
then $$a + b : a - b = x + y : x - y.$$

This is obtained by dividing the result of (i) by that of (ii).

V. *In a series of equal ratios (the magnitudes being all of the same kind), as any antecedent is to its consequent so is the sum of the antecedents to the sum of the consequents.*

Let each of the equal ratios $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}, \cdots$ be equal to k.

Then $$a = kx, b = ky, c = kz, \cdots;$$

$\therefore$, by addition,

$$a + b + c + \cdots = k(x + y + z + \cdots);$$

$$\therefore \frac{a + b + c + \cdots}{x + y + z + \cdots} = k = \frac{a}{x},$$

or, $$a : x = a + b + c + \cdots : x + y + z + \cdots.$$

VI. *A given straight line can be divided internally in a given ratio at one, and only one, point; and externally at one, and only one, point.*

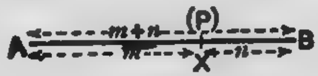


Fig.1.

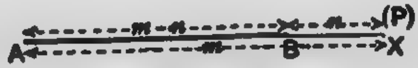


Fig.2.

Let AB be the given line, and $m : n$ the given ratio, m being greater than n.

Internal Division. (i) Divide AB (Fig. 1) into $m + n$ equal parts [*Prob.* 7]; and of these parts make AX to contain m; then XB must contain n.

Hence $\qquad AX : XB = m : n;$

that is, AB is divided *internally* at X in the given ratio.

(ii) Again, $\qquad AX : AB = m : m + n.$

Similarly, if P divides AB in the given ratio $m : n$,

$$AP : AB = m : m + n.$$

$$\therefore \frac{AX}{AB} = \frac{AP}{AB};$$

$$\therefore AX = AP.$$

Hence P and X coincide; that is, X is the only point which divides AB internally in the ratio $m : n$.

External Division. (i) Divide AB (Fig. 2) into $m - n$ equal parts; and in AB produced make AX to contain m such parts; then XB must contain n.

Hence $\qquad AX : XB = m : n;$

that is, AB is divided *externally* at X in the given ratio.

(ii) And it may be shewn, as above, that X is the only point which divides AB externally in the ratio $m : n$.

EXERCISES

1. Insert the missing terms in the following proportions:

(i) $3:7 = 15:(\quad)$;
(ii) $2\cdot5:(\quad) = 10:32$;
(iii) $(\quad):ac^2 = bc:bc^3$.

2. Correct the following statement:

£65 : 78 ft. = £25 : 30 ft.

3. If a straight line, 9·6″ in length, is divided *internally* in the ratio 5 : 7, calculate the lengths of the segments.

4. If a straight line 4·5 cm. in length is divided *externally* in the ratio 11 : 8, calculate the lengths of the segments.

5. AB is a straight line, 6·4 cm. in length, divided *internally* at X and *externally* at Y in the ratio 5 : 3; calculate the lengths of the segments, and shew that they satisfy the formula

$$\frac{2}{AB} = \frac{1}{AX} + \frac{1}{AY}.$$

6. If a straight line, a inches in length, is divided *internally* in the ratio $m:n$, shew that the lengths of the segments are respectively

$$\frac{m}{m+n}\cdot a \text{ inches}, \quad \frac{n}{m+n}\cdot a \text{ inches}.$$

7. If a straight line, a units in length, is divided *externally* in the ratio $m:n$, shew that the lengths of the segments are respectively

$$\frac{m}{m-n}\cdot a \text{ units}, \quad \frac{n}{m-n}\cdot a \text{ units}.$$

8. If $a:b = x:y$, and $b:c = y:z$, prove that $a:c = x:z$.

9. If $a:b = x:y$, shew that $a+b:a = x+y:x$.

10. If a, b, c are three proportionals, shew that $a:c = a^2:b^2$.

11. If two straight lines AB, CD are divided internally in the same ratio at X and Y respectively, shew that

(i) $AB:XB = CD:YD$;
(ii) $AB:AX = CD:CY$.

12. If a, b, c, d are four straight lines such that the rectangle contained by a and d is equal to that contained by b and c, prove that

$$a:b = c:d.$$

PROPORTIONAL DIVISION OF STRAIGHT LINES

THEOREM 46. [Euclid VI. 2]

A straight line drawn parallel to one side of a triangle cuts the other two sides, or those sides produced, proportionally.

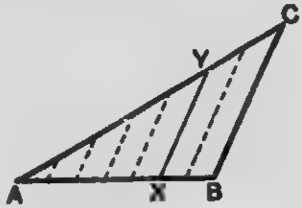


Fig. 1.

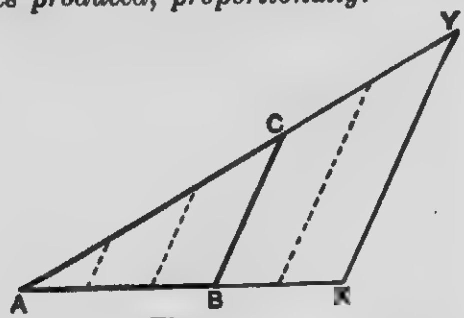


Fig. 2.

In the $\triangle ABC$, let XY, drawn par[l] to the side BC, cut AB, AC at X and Y, internally in Fig. 1, externally in Fig. 2.

It is required to prove in both cases that

$$AX : XB = AY : YC.$$

Proof.* Suppose X divides AB in the ratio $m : n$; that is, suppose

$$AX : XB = m : n;$$

so that, if AX is divided into m equal parts, then XB may be divided into n such equal parts.

Through the points of division in AX, XB let parallels be drawn to BC.

Then these parallels divide the segments AY, YC into parts which are all equal; *Theor.* 22.

and of these equal parts AY contains m,

and YC contains n;

hence $$AY : YC = m : n.$$

$$\therefore AX : XB = AY : YC.$$ Q.E.D.

* The proof given applies only to the case in which AX and XB are commensurable. The same is true of Theorems 48 and 49.

Conversely, *if a line cuts two sides of a triangle proportionally, it is parallel to the third side.*

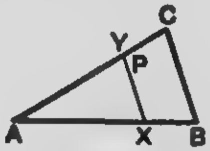


Fig. 1.

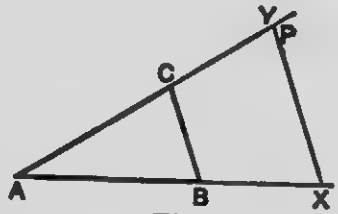


Fig. 2.

Conversely, let XY cut the sides AB, AC proportionally, so that

$$AX : XB = AY : YC.$$

It is required to prove that XY is parallel to BC.

Let XP be drawn through X par$^{\text{l}}$ to BC, to meet AC in P.

Then $\qquad AP : PC = AX : XB$;

but, by hypothesis, $\quad AY : YC = AX : XB.$

Thus AC is cut, internally in Fig. 1, and externally in Fig. 2, in the same ratio at P and Y.

Hence P coincides with Y, and consequently XP with XY.

Theor. VI, p. 208.

That is, $\qquad XY$ is par$^{\text{l}}$ to BC.

Q.E.D.

Corollary. If XY is parallel to BC, then

$$AX : AB = AY : AC.$$

For, taking Fig. 1, it may be shewn that

$$AX : AB = m : m + n;$$

and hence, by Theorem 22, that

$$AY : AC = m : m + n.$$

$$\therefore AX : AB = AY : AC.$$

Conversely, if $\quad AX : AB = AY : AC,$

it may be proved as above that XY is par$^{\text{l}}$ to BC.

THEOREM 47. [Euclid VI. 3 and 4]

If the vertical angle of a triangle is bisected internally or externally, the bisector divides the base internally or externally into segments which have the same ratio as the other sides of the triangle.

Conversely, if the base is divided internally or externally into segments proportional to the other sides of the triangle, the line joining the point of section to the vertex bisects the vertical angle internally or externally.

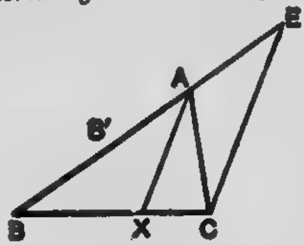


Fig. 1.

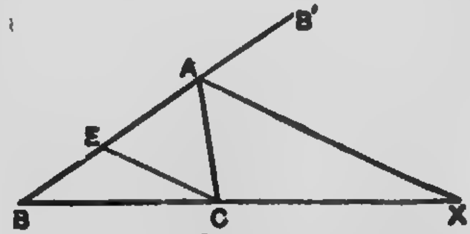


Fig. 2.

In the $\triangle ABC$, let AX bisect the $\angle BAC$, internally in Fig. 1, and externally in Fig. 2 ; that is, in the latter case, let AX bisect the exterior $\angle B'AC$.

It is required to prove in both cases that

$$BX : XC = BA : AC.$$

Let CE be drawn through C par^l to XA to meet BA (produced, if necessary) at E. In Fig. 1 let B' be taken in AB.

Proof. Because XA and CE are par^l,

$\therefore$, in both Figs., the $\angle B'AX$ = the int. opp. $\angle AEC$.

Also, the $\angle B'AX = \angle XAC$

= the alt. $\angle ACE$.

$\therefore$ the $\angle AEC$ = the $\angle ACE$.

$\therefore AC = AE$.

Again, because XA is par^l to CE, a side of the $\triangle BCE$,

$\therefore$, in both Figs., $BX : XC = BA \cdot AE$;

that is, $BX : XC = BA : AC$. Q.E.D.

Conversely, let BC be divided internally (Fig. 1) or externally (Fig. 2) at X, so that $BX : XC = BA : AC$.

It is required to prove that the $\angle\ B'AX =$ the $\angle\ XAC$.

Proof. For with the same construction as before,

because XA is par^l^ to CE, a side of the $\triangle\ BCE$,

$$\therefore BX : XC = BA : AE.$$

But, by hypothesis, $BX : XC = BA : AC$;

$$\therefore BA : AC = BA : AE;$$

$$\therefore AC = AE.$$

$$\therefore \text{the } \angle\ AEC = \text{the } \angle\ ACE$$

$$= \text{the alt. } \angle\ XAC.$$

And in both Figs.,

$$\text{the ext. } \angle\ B'AX = \text{the int. opp. } \angle\ AEC;$$

$$\therefore \text{the } \angle\ B'AX = \text{the } \angle\ XAC.$$

Q.E.D.

Definition

When a finite straight line is divided internally and externally into segments which have the same ratio, it is said to be cut **harmonically**.

Hence the following Corollary to Theorem 47.

The base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle;

for in each case the segments of the base are in the ratio of the other sides of the triangle.

EXERCISES ON THEOREM 46

(Numerical and Graphical)

1. On a base AB, 3·5″ in length, draw any triangle CAB; and from AB cut off AX 2·1″ long. Through X draw XY parallel to BC to meet AC at Y.

Measure AY, YC; and hence compare the ratios

$$\text{(i)}\ \frac{AX}{XB},\ \frac{AY}{YC};\quad \text{(ii)}\ \frac{AB}{AX},\ \frac{AC}{AY};\quad \text{(iii)}\ \frac{AB}{XB},\ \frac{AC}{YC}.$$

2. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides at X and Y.

(i) If $AB = 3·6''$, $AC = 2·4''$, and $AX = 2·1''$, calculate the length of AY.

(ii) If $AB = 2·0''$, $AC = 1·5''$, and $AY = 0·9''$, calculate the length of BX.

(iii) If X divides AB in the ratio 8 : 3, and if AC = 8·8 cm., find AY, YC.

3. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides produced at X and Y.

(i) If AB = 4·5 cm., AC = 3·5 cm., and AX = 7·2 cm., find by calculation and measurement the length of AY.

(ii) If X divides AB externally in the ratio 11 : 4, and if AC = 4·9 cm., find the segments of AC.

(Theoretical)

4. *Three parallel straight lines cut any two transversals proportionally.*

5. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

6. Two triangles ABC, DBC stand on the same side of the common base BC: and from any point E in BC lines are drawn parallel to BA, BD, meeting AC, DC in F and G. Shew that FG is parallel to AD.

7. In a triangle ABC a transversal is drawn to cut the sides BC, CA, AB (produced if necessary) at D, E, and F respectively. and it makes equal angles with AB and AC; prove that

$$BD : CD = BF : CE.$$

EXERCISES ON THEOREM 47

(Numerical and Graphical)

1. Draw a triangle ABC, making $a = 1{\cdot}5''$, $b = 2{\cdot}4''$, and $c = 3{\cdot}6''$. Bisect the angle A, internally and externally, by lines which meet BC and BC produced at X and Y.

Measure BX, XC; BY, YC; hence evaluate and compare the ratios

$$\frac{BX}{XC}, \frac{BY}{YC}, \frac{BA}{AC}.$$

2. In the triangle ABC, $a = 3{\cdot}5$ cm., $b = 5{\cdot}4$ cm., $c = 7{\cdot}2$ cm.; and the internal and external bisectors of the $\angle A$ meet BC at X and Y.

Calculate the lengths of the segments into which the base is divided at X and Y respectively; and verify your results graphically.

3. Frame constructions, based upon Theorem 47,
 (i) to trisect a straight line of given length;
 (ii) to divide a given line internally and externally in the ratio 3 : 2.

(Theoretical)

4. AD is a median of the triangle ABC; and the angles ADB, ADC are bisected by lines which meet AB, AC at E and F respectively. Shew that EF is parallel to BC.

5. $ABCD$ is a quadrilateral; shew that if the bisectors of the angles A and C meet on the diagonal BD, the bisectors of the angles B and D will meet on AC.

6. Employ Theorem 47 to shew that in any triangle
 (i) the internal bisectors of the three angles are concurrent;
 (ii) the external bisectors of two angles and the internal bisector of the third angle are concurrent.

7. If I is the in-centre of the triangle ABC, and if AI is produced to meet BC at X, shew that

$$AI : IX = AB + AC : BC.$$

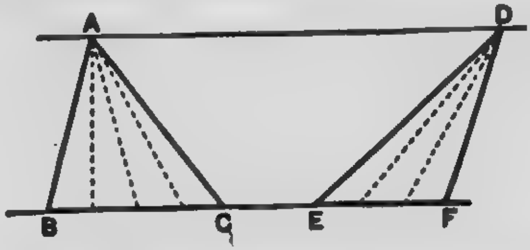
8. *Given the base of a triangle and the ratio of the other sides, find the locus of the vertex.*

9. Construct a triangle, having given the base, the ratio of the other sides, and the vertical angle.

PROPORTIONAL AREAS

THEOREM 48. [Euclid VI. 1]

The areas of triangles of equal altitude are to one another as their bases.



Let ABC, DEF be two triangles of equal altitude, standing on the bases BC, EF.

It is required to prove that

$$\text{the } \triangle ABC : \text{the } \triangle DEF = BC : EF.$$

Proof.* Let the triangles be placed so that the bases BC, EF are in the same st. line, and the triangles on the same side of the line.

Join AD;

then AD is par[l] to BF. *Def.* 2, *p.* 101.

Suppose the base BC : the base $EF = m : n$; so that, if BC is divided into m equal parts, then EF may be divided into n such equal parts, in each case by st. lines drawn from the vertex to the points of division.

Then the ▲ ABC, DEF are divided into triangles which stand on equal bases, and have the same altitude, and are therefore all equal.

And of these equal ▲, the $\triangle ABC$ contains m;

and the $\triangle DEF$ contains n.

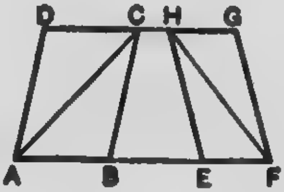
$\therefore$ the $\triangle ABC$: the $\triangle DEF = m : n$.

Hence the $\triangle ABC$: the $\triangle DEF = BC : EF$.

Q.E.D.

* See footnote on p. 210.

COROLLARY. *The areas of parallelograms of equal altitude are to one another as their bases.*



For let DB, EG be par^{ms} of the same altitude, standing on the bases AB, EF.

Join AC, HF.

Since the par^m DB = twice the $\triangle\, CAB$;
and the par^m EG = twice the $\triangle\, HEF$;
$\therefore$ the par^m DB : the par^m EG =
the $\triangle CAB$: the $\triangle\, HEF$ = AB : EF.

ALTERNATIVE PROOF OF THEOREM 48

Let p represent the altitude of each of the $\triangle$ ABC, DEF.
Then the area of the $\triangle\, ABC = \frac{1}{2}\cdot BC \times p$;
and the area of the $\triangle\, DEF = \frac{1}{2}\cdot EF \times p$.

$$\therefore \frac{\triangle\, ABC}{\triangle\, DEF} = \frac{\frac{1}{2}\cdot BC \times p}{\frac{1}{2}\cdot EF \times p} = \frac{BC}{EF}.$$

EXERCISES

(*Numerical*)

1. Of two triangles T_1, T_2 of equal altitude standing on bases of 6·3″ and 5·4″ T_1 contains $12\frac{1}{2}$ sq. inches. Find the area of T_2.

2. The areas of two triangles of equal altitude have the ratio 24 : 17; if the base of the first is 4·2 cm., find the base of the other.

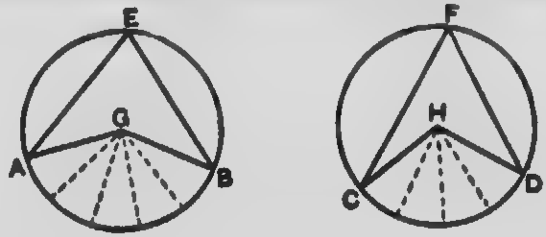
3. Two triangles lying between the same parallels have bases of 16·20 m. and 20·70 m.; find to the nearest square cm. the area of the second triangle, if that of the first is 50·1204 sq. m.

4. Two parallelograms whose areas are in the ratio 2·1 : 3·5 lie between the same parallels. If the base of the first is 6·6″ in length, find the base of the second.

5. Two triangular fields lie on opposite sides of a common base; and their altitudes with respect to it are 4·20 chains and 3·71 chains. If the first field contains 18 acres, find the acreage of the other.

THEOREM 49. [Euclid VI. 33]

In equal circles, angles, whether at the centres or circumferences, have the same ratio as the arcs on which they stand.



Let ABE, CDF be equal circles; and let the ∠s AGB, CHD, and also the ∠s AEB, CFD stand on the arcs AB, CD.

It is required to prove that

(i) *the* $\angle AGB$: *the* $\angle CHD$ = *the arc* AB : *the arc* CD;

(ii) *the* $\angle AEB$: *the* $\angle CFD$ = *the arc* AB : *the arc* CD.

Proof.* Suppose the arc AB : the arc $CD = m : n$;
so that, if the arc AB is divided into m equal parts, then the arc CD may be divided into n such equal parts, in each case by radii drawn to the points of division.

Then the ∠s AGB, CHD, in equal circles, are divided into angles which stand on equal arcs, and are therefore all equal.

And of these equal angles the $\angle AGB$ contains m,
and the $\angle CHD$ contains n;

∴ the $\angle AGB$: the $\angle CHD = m : n$.

Hence the $\angle AGB$: the $\angle CHD$ = the arc AB: the arc CD.

And since the $\angle AEB$ = one half of the $\angle AGB$; *Theor.* 39
and the $\angle CFD$ = one half of the $\angle CHD$;

∴ the $\angle AEB$: the $\angle CFD$ = the arc AB : the arc CD.

Q.E.D.

COROLLARY. Since *in equal circles, sectors which have equal angles are equal* [*p.* 147, *E*], it may be proved as above that *the sector* AGB : *the sector* CHD = *the arc* AB : *the arc* CD.

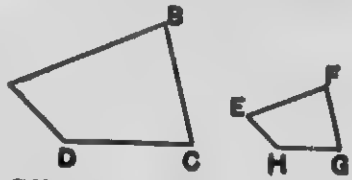
* See footnote on p. 210.

SIMILAR FIGURES

1. Two rectilineal figures are said to be **equiangular** to one another when the angles of the first, taken in order, are equal respectively to those of the second, taken in order.

2. Rectilineal figures are said to be **similar** when they are equiangular to one another, and also have their corresponding sides proportional.

Thus the two quadrilaterals $ABCD$, $EFGH$ are similar if the angles at A, B, C, D are respectively equal to those at E, F, G, H, and if also



$$AB : EF = BC : FG = CD : GH = DA : HE.$$

3. Similar figures are said to be **similarly described** with regard to two sides, when these sides correspond.

NOTE ON SIMILAR FIGURES

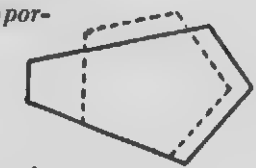
Similar figures may be described as having the *same shape*. For this, the figures must satisfy *two* conditions:

(i) *they must have their angles equal each to each, taken in order;*
(ii) *their corresponding sides must be proportional.*

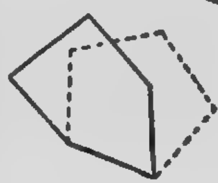
In the case of *triangles* we shall learn that these conditions are not independent, for each follows from the other: thus

(i) if the triangles are *equiangular to one another*, Theorem 50 proves *that their corresponding sides are proportional;*

(ii) if the triangles have their *sides proportional*, Theorem 51 proves *that they are equiangular to one another.*



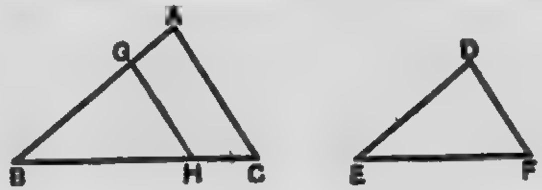
On the other hand, the first diagram in the margin shews two figures which are equiangular to one another, but which clearly have not their sides proportional; while the figures in the second diagram have their sides proportional, but are not equiangular to one another.



SIMILAR TRIANGLES

THEOREM 50. [Euclid VI. 4]

If two triangles are equiangular to one another, their corresponding sides are proportional, and the triangles are similar.



Let the $\triangle$ ABC, DEF have the $\angle$ A, B, and C respectively equal to the $\angle$ D, E, and F.

It is required to prove that

$$AB : DE = BC : EF = CA : FD.$$

Proof. Apply the $\triangle$ DEF to the $\triangle$ ABC, so that E falls on B, and EF along BC;

then since the $\angle E$ = the $\angle B$, ED will fall along BA.

Let D and F fall at G and H respectively; so that GBH represents the $\triangle$ DEF in its new position.

Now, by hypothesis, the $\angle D$ = the $\angle A$;

that is, the ext. $\angle BGH$ = the int. opp. $\angle BAC$;

$\therefore$ GH is par^l to AC.

Hence $\quad BA : BG = BC : BH$; *Theor.* 46, *Cor.*

that is, $\quad AB : DE = BC : EF$.

Similarly, by applying the $\triangle$ DEF to the $\triangle$ ABC, so that F falls on C, and FE, FD along CB, CA, it may be shewn that

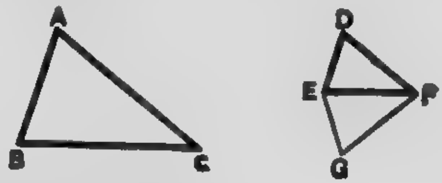
$$BC : EF = CA : FD.$$

Hence $\quad AB : DE = BC : EF = CA : FD$,

and so the triangles are similar (see p. 219). Q.E.D.

THEOREM 51. [Euclid VI. 5]

If two triangles have their sides proportional when taken in order, the triangles are equiangular to one another, and the triangles are similar.



In the $\triangle$ ABC, DEF, let

$$AB : DE = BC : EF = CA : FD.$$

It is required to prove that the $\triangle$ ABC, DEF *are equiangular to one another.*

At E in FE make the $\angle FEG$ equal to the $\angle B$; and at F in EF make the $\angle EFG$ equal to the $\angle C$.

$\therefore$ the remaining $\angle EGF$ = remaining $\angle A$.

Proof. Since the $\triangle$ ABC, GEF are equiangular to one another,

$\therefore AB : GE = BC : EF.$ *Theor.* 50.

But, by hypothesis, $AB : DE = BC : EF$;

$\therefore AB : GE = AB : DE.$

$\therefore GE = DE.$

Similarly $GF = DF$.

Then in the $\triangle$ GEF, DEF,

because $\left\{ \begin{array}{l} GE = DE,\ GF = DF, \\ \text{and } EF \text{ is common;} \end{array} \right.$

$\therefore$ the triangles are identically equal; *Theor.* 7.

$\therefore$ the $\angle DEF$ = the $\angle GEF$ = the $\angle B$;

and the $\angle DFE$ = the $\angle GFE$ = the $\angle C$.

$\therefore$ the remaining $\angle D$ = the remaining $\angle A$;

that is, the $\triangle DEF$ is equiangular to the $\triangle ABC$.

Hence the triangles are similar (see p. 219.) Q.E.D.

EXERCISES ON SIMILAR TRIANGLES

(Numerical and graphical. The results are to be obtained by calculation and checked graphically)

1. In a triangle ABC, XY is drawn parallel to BC, cutting the other sides at X and Y:

(i) If $AB = 2{\cdot}5''$, $AC = 2{\cdot}0''$, $AX = 1{\cdot}5''$; find AY.
(ii) If $AB = 3{\cdot}5''$, $AC = 2{\cdot}1''$, $AY = 1{\cdot}2''$; find AX.
(iii) If $AB = 4{\cdot}2$ cm., $AX = 3{\cdot}6$ cm., $AY = 6{\cdot}6$ cm.; find AC.

2. In the figure of the last example:

(i) If $AB = 2{\cdot}4''$, $BC = 3{\cdot}6''$, $AX = 1{\cdot}4''$; find XY.
(ii) If $BC = 7{\cdot}7$ cm., $XY = 5{\cdot}5$ cm., $AX = 4{\cdot}5$ cm.; find AB.

3. In the triangle ABC, $a = 3{\cdot}0''$, $b = 3{\cdot}6''$, $c = 4{\cdot}2''$; and QR, drawn parallel to AC, measures $3{\cdot}0''$. Find the remaining sides of the triangle QBR.

4. ABC is a triangle in which $a = 8$ cm., $b = 7$ cm., and $c = 10$ cm. In AB a point P is taken 4 cm. from A, and PQ is drawn parallel to BC. Find the lengths of PQ and QC.

5. The sides of a triangular field are 400 yards, 350 yards, and 300 yards respectively. In a plan of the field the greatest side measures $2{\cdot}4''$; find the lengths of the other sides.

6. XY is drawn parallel to BC, the base of the triangle ABC. If $AX = 8\frac{1}{3}$ ft., $XY = 3\frac{3}{4}$ ft., $AY = 6$ ft. 2 in., and $XB = 4\frac{1}{6}$ ft.; calculate the sides of the triangle ABC.

7. The triangle ABC is right-angled at C; and from P, a point in the hypotenuse, PQ is drawn parallel to AC.

If $AC = 1\frac{1}{4}''$, $BC = 3''$, and $PQ = \frac{1}{2}''$; find BQ, BP, and AP.

8. In a triangle ABC, AD is the perpendicular from A on BC; and through X, a point in AD, a parallel is drawn to BC, meeting the other sides in P, Q.

If $BC = 9$ cm., $AD = 8$ cm., $DX = 3$ cm.; find PQ.

9. In the triangle ABC, $a = 2{\cdot}0$ cm., $b = 3{\cdot}5$ cm., $c = 4{\cdot}5$ cm. BD and CE are drawn from the ends of the base to the opposite sides, and they intersect in P.

If $$EP : PC = DP : PB = 2 : 5,$$
find the lengths of ED, AD, and DC.

EXERCISES ON SIMILAR TRIANGLES

(*Theoretical*)

1. Shew that the straight line which joins the middle points of two sides of a triangle is
   (i) parallel to the third side; (ii) one-half the third side.

2. In the trapezium $ABCD$, AB is parallel to DC, and the diagonals intersect at O: shew that

$$OA : OC = OB : OD.$$

If $AB = 2DC$, shew that O is a point of trisection on both diagonals.

3. If three concurrent straight lines are cut by two parallel transversals in A, B, C, and P, Q, R respectively; prove that

$$AB : BC = PQ : QR.$$

4. $ABCD$ is a parallelogram, and from D a straight line is drawn to cut AB at E, and CB produced at F. In this figure name three triangles which are equiangular to one another; and shew that

$$DA : AE = FB : BE = FC : CD.$$

5. In the side AC of a triangle ABC any point D is taken: shew that if AD, DC, AB, BC are bisected in E, F, G, H respectively then EG is equal to HF.

6. AB and CD are two parallel straight lines; E is the mid-point of CD; AC and BE meet at F, and AE and BD meet at G: shew that FG is parallel to AB.

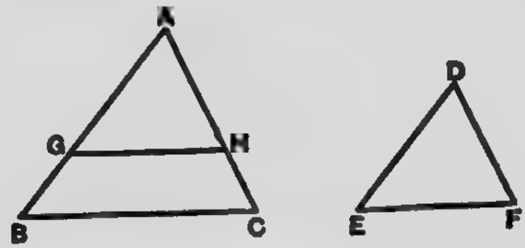
7. AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference in C and the tangent at B and D; shew that
   (i) the $\triangle$ CAB, BAD are equiangular to one another;
   (ii) AC, AB, AD are three proportionals;
   (iii) the rect. AC, AD is constant for all positions of AD.

8. If through any point X within a circle two chords AB, CD are drawn, and AC, BD joined; shew that
   (i) the $\triangle$ AXC, DXB are equiangular to one another;
   (ii) $AX : DX = XC : XB$.

9. If from an external point X a tangent XT and a secant XAB are drawn to a circle, and AT, TB joined; shew that
   (i) the $\triangle$ AXT, TXB are equiangular to one another;
   (ii) $XA : XT = XT : XB$.

THEOREM 52. [Euclid VI. 6]

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles are similar.



In the $\triangle$ ABC, DEF, let the $\angle A$ = the $\angle D$,

and let $AB : DE = AC : DF$.

It is required to prove that the $\triangle$ ABC, DEF *are similar.*

Proof. Apply the $\triangle DEF$ to the $\triangle ABC$, so that D falls on A, and DE along AB; then

because the $\angle EDF$ = the $\angle BAC$, DF must fall along AC.

Let G and H be the points at which E and F fall respectively; so that AGH represents the $\triangle DEF$ in its new position.

Now, by hypothesis, $AB : DE = AC : DF$;

that is, $AB : AG = AC : AH$;

hence GH is par[l] to BC. *Theor.* 46, *Cor.*

$\therefore$ the ext. $\angle AGH$, namely the $\angle E$, = the int. opp. $\angle ABC$; and the ext. $\angle AHG$, namely the $\angle F$, = the int. opp. $\angle ACB$.

Hence the $\triangle$ ABC, DEF are equiangular to one another,

hence, the $\triangle$ ABC, DEF are similar. *Theor.* 50.

Q.E.D.

* THEOREM 53. [Euclid VI. 7]

If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in one proportional to the corresponding sides of the other, then the third angles are either equal or supplementary; and in the former case the triangles are similar.

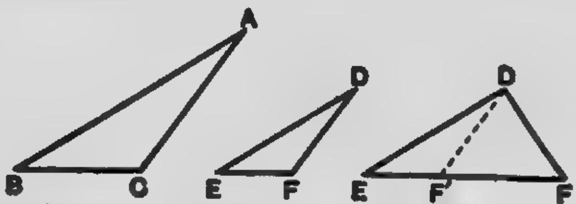


Fig. 1. Fig. 2. Fig. 3.

In the ▲ ABC, DEF, let the $\angle B$ = the $\angle E$; and let

$$AB : DE = AC : DF.$$

It is required to prove that

either the $\angle C$ = the $\angle F$ [as in Figs. 1 and 2];

or the $\angle C$ = *the supplement of the* $\angle F$ [Figs. 1 and 3].

Proof. (i) If the $\angle A$ = the $\angle D$ [Figs. 1 and 2],

then the $\angle C$ = the $\angle F$; *Theor.* 16.

and the ▲ are equiangular, and therefore similar.

(ii) If the $\angle A$ is not equal to the $\angle D$ [Figs. 1 and 3],

let the $\angle EDF'$ = the $\angle A$.

Then the ▲ ABC, DEF' are equiangular to one another;

$$\therefore AB : DE = AC : DF'.$$

But $AB : DE = AC : DF$; (*Hypothesis*)

$$\therefore AC : DF' = AC : DF.$$

$$\therefore DF' = DF.$$

$\therefore$ the $\angle DFF'$ = the $\angle DF'F$.

= the supplement of the $\angle DF'E$

= the supplement of the $\angle C$.

Q.E.D.

EXERCISES ON SIMILAR TRIANGLES

(*Theoretical*)

1. In a triangle ABC, prove that any straight line parallel to the base BC and intercepted by the other two sides is bisected by the median drawn from the vertex A.

2. Two triangles ABC, $A'B'C'$ are equiangular to one another;
if p, p' denote the perpendiculars from A, A' to the opp. sides
R, R' . . . . . . circum-radii;
r, r' . . . . in-radii;

prove that each of the ratios $\frac{p}{p'}$, $\frac{R}{R'}$, $\frac{r}{r'}$ is equal to the ratio of any pair of corresponding sides.

3. Prove that the radius of the circle which passes through the mid-points of the sides of a triangle is half the circum-radius.

4. If two straight lines AB, CD intersect at X, so that

$$XA : XC = XD : XB;$$

(i) shew by Theorem 52 that the ▲ AXD, CXP are similar;
(ii) hence prove the points A, D, B, C concyclic.

5. A, B, C are three collinear points, and from B and C two parallel lines BP, CQ are drawn in the same sense, so that

$$PB : QC = AB : AC,$$

shew by Theorem 52 that the points A, P, Q are collinear.

6. If in two triangles ABC, $A'B'C'$, the $\angle B$ = the $\angle B'$, and $\frac{c}{c'} = \frac{b}{b'}$; what conclusion may be drawn?

Shew by diagrams how this conclusion is affected, if it is also given that

(i) c is less than b,
(ii) c is equal to b,
(iii) c is greater than b.

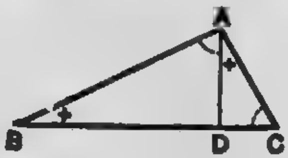
7. $ABCD$ is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S: shew that RS is parallel to AD.

8. In a triangle ABC the bisector of the vertical angle A meets the base at D and the circumference of the circum-circle at E; if EC is joined, shew that the triangles BAD, EAC are similar; and hence prove that

$$AB \cdot AC = AE \cdot AD.$$

Theorem 54. [Euclid VI. 8]

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let BAC be a triangle right-angled at A, and let AD be drawn perp. to BC.

It is required to prove that the $\triangle$s BDA, ADC *are similar to the* $\triangle$ BAC *and to one another.*

In the $\triangle$s BDA, BAC

the $\angle BDA$ = the $\angle BAC$, being rt. angles,

and the $\angle B$ is common to both;

$\therefore$ the remaining $\angle BAD$ = the remaining $\angle BCA$; *Theor.* 16.

hence the $\triangle BDA$ is equiangular to the $\triangle BAC$;

$\therefore$ their corresponding sides are proportional;

$\therefore$ the $\triangle$s BDA, BAC are similar.

Similarly the $\triangle$s ADC, BAC may be proved similar.

Hence the $\triangle$s BDA, ADC, being equiangular to the $\triangle BAC$, are equiangular and hence similar to each other. Q.E.D

Corollary. (i) Because the $\triangle$s DBA, DAC are similar,

$$\therefore DB : DA = DA : DC;$$

that is, DA is a mean proportional between DB and DC;

and hence $$DA^2 = DB \cdot DC.$$

(ii) Because the $\triangle$s BCA, BAD are similar,

$$\therefore BC : BA = BA : BD;$$

hence $$BA^2 = BC \cdot BD.$$

(iii) Because the $\triangle$s CBA, CAD are similar,

$$\therefore CB : CA = CA : CD;$$

hence $$CA^2 = CB \cdot CD.$$

EXERCISES

(*Miscellaneous Examples on Theorems 50–54*)

1. ABC is an equilateral triangle of which each side $= a$. In BC, produced both ways, two points P and Q are taken, such that $BP = CQ = a$, and AP, AQ are joined. Shew that

(i) $PQ : PA = PA : PB$.

(ii) $PA^2 = 3a^2$.

2. ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC: if AB, AC measure respectively 4″ and 3″, shew that the segments of the hypotenuse are 3·2″ and 1·8″.

3. ABC is a triangle right-angled at A, and a perpendicular AD is drawn to the hypotenuse BC; shew (i) by Theorem 25, (ii) by Theorem 54 that

$$BC \cdot AD = AB \cdot AC.$$

4. ABC is a triangle right-angled at A, and AC' is drawn perpendicular to the hypotenuse, also $C'A'$ is drawn parallel to CA. If $AC = 15$ cm., and $AB = 20$ cm., shew that $AC' = 12$ cm., and $C'A' = 9{\cdot}6$ cm.

5. At the extremities of a diameter of a circle, whose centre is C and radius r, tangents are drawn: these are cut in Q and R by any third tangent whose point of contact is P. Shew that

(i) QR subtends a right angle at C;

(ii) $PQ \cdot PR = r^2$.

6. Two circles of radii r and r' respectively have external contact at A, and a common tangent touches them at P and Q. Shew that

(i) PQ subtends a right angle at A; [Ex. 9. p. 182.]

(ii) $PQ^2 = 4rr'$.

[Produce PA, QA to meet the circumferences at X and Y, and prove the triangles PAY, XAQ right-angled and similar.]

7. Two circles touch one another externally at A, and a common tangent PQ is produced to meet the line of centres at S. Shew that, if PA, AQ are joined,

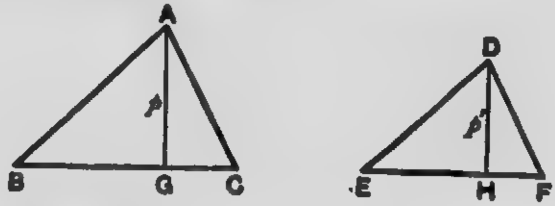
(i) the triangles SAP, SQA are similar;

(ii) $SA^2 = SP \cdot SQ$.

8. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D: shew that if BC, BD are joined, then $BC : BA = BA : BD$.

THEOREM 55. [Euclid VI. 19]

The areas of similar triangles are proportional to the squares on corresponding sides.



Let ABC, DEF be similar triangles, in which BC and EF are corresponding sides.

It is required to prove that

$$\text{the } \triangle ABC : \text{the } \triangle DEF = BC^2 : EF^2.$$

Let AG and DH be drawn perp. to BC, EF respectively; and denote these perp^s. by p and p'.

Proof. The $\triangle ABC = \frac{1}{2} BC \cdot p$; the $\triangle DEF = \frac{1}{2} EF \cdot p'$.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot p}{EF \cdot p'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

But since the $\angle B =$ the $\angle E$, from the similar $\triangle$s ABC, DEF,

and the $\angle G =$ the $\angle H$, being right angles;

$\therefore$ the $\triangle$s ABG, DEH are equiangular to one another, *Theor.* 16.

$$\therefore \frac{p}{p'} = \frac{AB}{DE} \qquad \textit{Theor. } 50.$$

$$= \frac{BC}{EF}, \text{ from the similar } \triangle\text{s } ABC, DEF.$$

Substituting for $\frac{p}{p'}$ in (i),

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot BC}{EF \cdot EF} = \frac{BC^2}{EF^2};$$

or, the $\triangle ABC$: the $\triangle DEF = BC^2 : EF^2$. Q.E.D.

EXERCISES ON THE AREAS OF SIMILAR TRIANGLES

(Numerical and Graphical)

1. In any triangle ABC, the sides AB, AC are cut by a line XY drawn parallel to BC. If AX is one-third of AB, what part is the triangle AXY of the triangle ABC?

2. Two corresponding sides of similar triangles are 3 ft. 6 in. and 2 ft. 4 in. respectively. If the area of the greater triangle is 45 sq. ft., find that of the smaller.

3. The area of the triangle ABC is 25·6 sq. cm., and XY, drawn parallel to BC, cuts AB in the ratio 5 : 3. Find the area of the triangle AXY.

4. Two similar triangles have areas of 392 sq. cm. and 200 sq. cm. respectively; find the ratio of any pair of corresponding sides.

5. ABC and XYZ are two similar triangles whose areas are respectively 32 sq. in. and 60·5 sq. in. If $XY = 7{\cdot}7''$, find the length of the corresponding side AB.

6. Shew how to draw a straight line XY parallel to BC the base of a triangle ABC, so that the area of the triangle AXY may be nine-sixteenths of that of the triangle ABC.

(Theoretical)

7. ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC; shew that

$$\triangle BAD : \triangle ACD = BA^2 : AC^2.$$

8. A trapezium $ABCD$ has its sides AB, CD parallel, and its diagonals intersect at O. If AB is double of CD, find the ratio of the triangle AOB to the triangle COD.

9. If two triangles have one angle of one equal to one angle of the other, their areas are proportional to the rectangles contained by the sides about the equal angles.

10. Prove that the areas of similar triangles have the same ratio as the *squares of*

(i) corresponding altitudes;
(ii) corresponding medians;
(iii) the radii of their in-circles;
(iv) the radii of their circum-circles.

RECTANGLES IN CONNECTION WITH CIRCLES

THEOREM 56. [Euclid III. 35 and 36]

If any two chords of a circle cut one another internally or externally, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

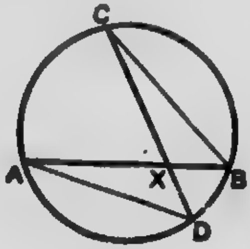


Fig. 1.

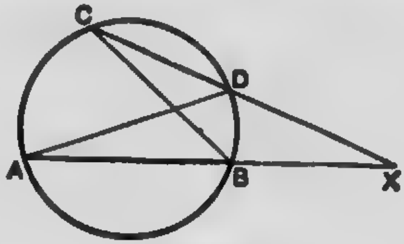


Fig. 2.

In the ⊙ ABC, let the chords AB, CD cut one another at X, internally in Fig. 1, and externally in Fig. 2.

It is required to prove in both cases that

the rect. XA, XB = *the rect.* XC, XD.

Join AD, BC.

Proof. In the ▲ AXD, CXB,

the ∠ AXD = the ∠ CXB, being opp. vert. ∠s in Fig. 1, and the same angle in Fig. 2;

and the ∠ A = the ∠ C, being ∠s at the ○ce standing on the same arc BD;

∴ the remaining angles are equal; *Theor.* 16.

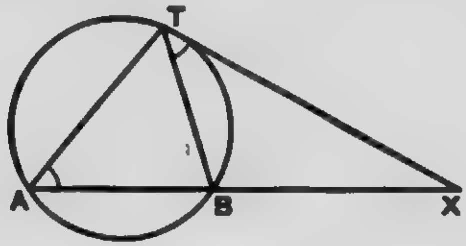
hence the ▲ AXD, CXB are equiangular,

$$\therefore \frac{XA}{XC} = \frac{XD}{XB};$$

$$\therefore XA \cdot XB = XC \cdot XD;$$

that is, the rect. XA, XB = the rect. XC, XD. Q.E.D.

Corollary. *If from an external point a secant and a tangent are drawn to a circle, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.*



Let XBA be a secant, and XT a tangent drawn to the $\odot\ ABT$ from the point X.

It is required to prove that $XA \cdot XB = XT^2$.

Proof. Join AT, BT.

Then

because $\left\{\begin{array}{l}\text{the } \angle XAT = \text{the } \angle XTB, \quad \textit{Theor. } 45. \\ \text{the } \angle TXA \text{ is common,} \\ \text{the third angles are equal,} \quad \textit{Theor. } 16.\end{array}\right.$

the $\triangle\ XAT$, XBT are similar. *Theor.* 50.

$$\therefore \frac{XA}{XT} = \frac{XT}{XB}.$$

$\therefore$ rect. XA, XB = sq. on XT. Q.E.D.

EXERCISES ON THEOREM 56

(Theoretical)

1. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse; shew that

$$AD \cdot DB = CD^2.$$

2. If two circles intersect, and through any point X in their common chord two chords AB, CD are drawn, one in each circle, shew that

$$AX \cdot XB = CX \cdot XD.$$

3. Deduce from Theorem 56 that the tangents drawn to a circle from any external point are equal.

4. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

5. If a common tangent PQ is drawn to two circles which cut at A and B, shew that AB produced bisects PQ.

6. If two straight lines AB, CD intersect at X so that $AX \cdot XB = CX \cdot XD$, deduce from Theorem 56 (by *reductio ad absurdum*) that the points A, B, C, D are concyclic.

7. In the triangle ABC, perpendiculars AP, BQ are drawn from A and B to the opposite sides, and intersect at O; shew that

$$AO \cdot OP = BO \cdot OQ.$$

8. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse; shew that

$$AB \cdot AD = AC^2.$$

9. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences; shew that

$$CA \cdot AE = DA \cdot AF.$$

10. If from any external point P two tangents are drawn to a given circle whose centre is O and radius r; and if OP meets the chord of contact at Q, shew that

$$OP \cdot OQ = r^2.$$

11. AB is a fixed diameter of a circle, and CD is perpendicular to AB (or AB produced); if *any* straight line is drawn from A to cut CD at P and the circle at Q, shew that

$$AP \cdot AQ = \text{constant}.$$

EXERCISES ON THEOREM 56

(Miscellaneous)

1. The chord of an arc of a circle $= 2c$, the height of the arc $= h$, the radius $= r$. Shew by Theorem 56 that

$$h(2r - h) = c^2.$$

Hence find the diameter of a circle in which a chord 24″ long cuts off a segment 8″ in height.

2. The radius of a circular arch is 25 feet, and its height is 18 feet; find the span of the arch.

If the height is reduced by 8 feet, the radius remaining the same, by how much will the span be reduced?

Check your calculated results graphically by a diagram in which 1″ represents 10 feet.

3. Employ the equation $h(2r - h) = c^2$ to find the height of an arc whose chord is 16 cm., and radius 17 cm.

Explain the double result geometrically.

4. If d denotes the shortest distance from an external point to a circle, and t the length of the tangent from the same point, shew by Theorem 56 that

$$d(d + 2r) = t^2.$$

Hence find the diameter of the circle when $d = 1{\cdot}2''$, and $t = 2{\cdot}4''$; and verify your result graphically.

5. If the horizon visible to an observer on a cliff 330 feet above the sea-level is $22\frac{1}{4}$ miles distant, find roughly the diameter of the earth.

Hence find the approximate distance at which a bright light raised 66 feet above the sea is visible at the sea-level.

6. If h is the height of an arc of radius r, and b the chord of half the arc, prove that $b^2 = 2rh$.

7. A semi-circle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P; shew that

$$AB^2 = AC \cdot AP + BD \cdot BP.$$

8. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn; if the common chord is produced to meet the tangents at G and H, shew that

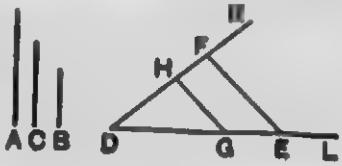
$$GH^2 = AE^2 + BC^2.$$

PROBLEMS

Problem 33

To find the **fourth proportional** *to three given straight lines.*

Let A, B, C be the three given st. lines, to which the fourth proportional is required.



Construction. Draw two st. lines DL, DK of indefinite length, containing any angle.

From DL cut off DG equal to A, and GE equal to B; and from DK cut off DH equal to C.

Join GH. Through E draw EF par^l to GH.

Then HF is the fourth proportional to A, B, C.

Proof. Because GH is par^l to EF, a side of the $\triangle DEF$;

$$\therefore DG : GE = DH : HF.$$

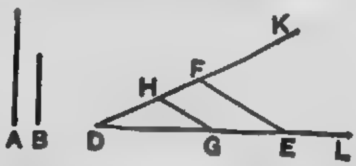
That is, $A : B = C : HF$.

Then HF is the fourth proportional to A, B, C.

Problem 34

To find the **third proportional** *to two given straight lines.*

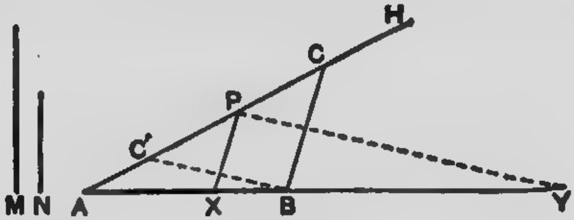
Let A, B be the two lines to which the third proportional is required.



This problem is that special case of Problem 36 in which $C = B$. (See 5, p. 205.) The solution given above applies to it.

Problem 35

To divide a given straight line **internally** *and* **externally** *in a* **given ratio.**



Let AB be the st. line to be divided internally and externally in the ratio $M : N$.

Construction. At A make any angle BAH with AB.

From AH cut off AP equal to M.

From PH and PA cut off PC and PC', each equal to N.

Join BC, BC'.

Through P draw PX par[l] to BC, and PY par[l] to BC'.

Then AB is divided internally at X, and externally at Y in the ratio $M : N$.

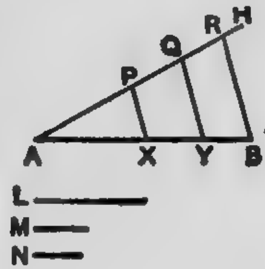
Proof. (i) Because PX is par[l] to BC, a side of the $\triangle ABC$,

$$\therefore AX : XB = AP : PC = M : N.$$

(ii) Because PY is par[l] to BC, a side of the $\triangle ABC'$,

$$\therefore AY : YB = AP : PC' = M : N.$$

Corollary. By a similar process a st. line AB may be divided internally into segments proportional to three lines.



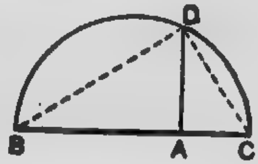
Construction. Draw AH, and from it cut off AP, PQ, QR equal respectively to L, M, N. Join RB; and through P and Q draw PX, QY par[l] to BR.

Then evidently

$$AX : L = XY : M = YB : N.$$

Problem 36

To find the **mean proportional** *between two given straight lines.*



Let AB, AC be the two given st. lines.

Construction. Place AB, AC in a straight line, and in opposite senses; and on BC describe the semi-circle BDC.

From A draw AD at rt. angles to BC, to cut the $\bigcirc^{ce}$ at D.

Then AD is the mean proportional between AB and AC.

Proof. Join BD, DC.

Now the $\angle\, BDC$, being in a semi-circle, is a rt. angle.

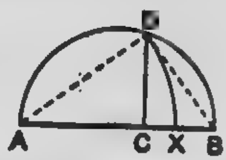
And in the right-angled $\triangle\, BDC$, DA is perp. to BC,

$\therefore$ the $\blacktriangle$ ABD, ADC are similar; *Theor.* 54.

$$\therefore AB : AD = AD : AC;$$

that is, AD is the mean proportional between AB and AC.

Note. If the given lines AB, AC are placed in the same sense, the mean proportional between them may be cut off from AB by the following useful construction.



On AB draw a semi-circle; and from C draw CD perp. to AB to cut the $\bigcirc^{ce}$ at D. From AB cut off AX equal to AD.

Then AX is the mean proportional between AB and AC.

For the $\blacktriangle$ ABD, ADC are similar, *Theor.* 54.

$$\therefore AB : AD = AD : AC;$$

that is, $$AB : AX = AX : AC.$$

GRAPHICAL EVALUATION OF A QUADRATIC SURD

Example. Find the approximate value of (i) $\sqrt{5}$, (ii) $\sqrt{21}$.

(i) $\sqrt{5} = \sqrt{5 \times 1}$. Hence take AB, AC respectively to represent 5 and 1 in terms of any convenient unit, and find AD, the mean proportional between them.

Then $$AD^2 = AB \cdot AC \qquad \text{III, p. 206.}$$
$$= 5 \times 1 = 5.$$
$$\therefore AD = \sqrt{5}.$$

By measuring AD, the value of $\sqrt{5}$ is roughly found to be 2·24.

(ii) $\sqrt{21} = \sqrt{7 \times 3}$. Here take AB, AC equal to 7 cm. and 3 cm. respectively, and proceed as before.

Note. Factors should be chosen so as to give convenient lengths for AB, AC.

e.g. $\sqrt{23} = \sqrt{2{\cdot}3 \times 10}$; $\sqrt{11} = \sqrt{2{\cdot}2 \times 5}$.

EXERCISES

1. Find graphically, testing your results by arithmetic:
   (i) The 4th proportional to 2·4″, 1·5″, 1·6″.
   (ii) The 3rd proportional to 2·5″ and 1·5″.
   (iii) The mean proportional between 7·2 cm. and 5·0 cm.

2. Divide a line, 2·0″ in length, internally and externally in the ratio 7 : 3; and in each case measure and calculate the segments.

3. Obtain graphically the unknown term in the following statements of proportion; and check your result by arithmetic:
   (i) $1{\cdot}25 : x = 1{\cdot}0 : 1{\cdot}6$. [Take 1″ as the unit of length.]
   (ii) $x : 4{\cdot}2 = 4{\cdot}2 : 6{\cdot}3$. [Take 1 cm. as the unit of length.]
   (iii) $x : 16 = 25 : x$. [Let 1″ represent 10.]

4. Divide a line, 7·2 cm. in length, into three parts proportional to the numbers 2, 3, 4. Measure and calculate these parts.

5. Divide a line, 3·9″ in length, into three parts, so that the second = $\frac{2}{3}$ of the first, and the third = $\frac{3}{4}$ of the second.

6. On a side of 1·5″ draw a rectangle equal in area to a square on a side of 2″. Measure the other side of the rectangle.

7. Find graphically the approximate values of

   (i) $\sqrt{3}$; (ii) $\sqrt{10}$; (iii) $\sqrt{\frac{14}{5}}$.

8. Determine geometrically the approximate values of the following expressions, verifying each drawing arithmetically:

$$\text{(i)}\ \frac{3{\cdot}5 \times 2{\cdot}4}{2{\cdot}8};\quad \text{(ii)}\ \frac{6{\cdot}84}{2{\cdot}13};\quad \text{(iii)}\ \frac{2{\cdot}71 \times 1{\cdot}26}{1{\cdot}51}.$$

9. Draw a triangle ABC from each of the following sets of data, and in each case calculate and measure the lengths of the sides:

(i) The perimeter = 4·8″; and $a:3 = b:4 = c:5$.
(ii) The perimeter = 11·1 cm.; and $a = \frac{5}{6}b$, $b = \frac{4}{5}c$.
(iii) The perimeter = 11·8 cm.; and $A:1 = B:2 = C:3$.
(iv) $a = 4{\cdot}0''$; $A = 90°$; and $b:c = 5:3$.

10. A field is represented in a plan by a triangle ABC, in which $a = 8$ cm., $b = 5{\cdot}6$ cm., $c = 6{\cdot}4$ cm. If the greatest side of the field is 200 metres, find the lengths of the other sides.

A fence, run across the field, is represented in the plan by a line PQ parallel to BC drawn from a point P in AB distant 4·0 cm. from A. Find the length of the fence.

11. A man 6 feet in height, standing 15 feet from a lamp-post, observes that his shadow cast by the light is 5 feet in length; how high is the light, and how long would his shadow be if he were to approach 8 feet nearer to the post?

12. To find the width of a canal a rod is fixed vertically on the bank so as to shew 4½ feet of its length. The observer, whose eye is 5 ft. 8 in. above the ground, retires at right angles from the canal until he sees the top of the rod in a line with the further bank. If his distance from the canal is now 20 feet, what is its width?

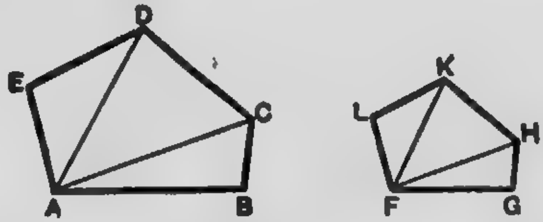
13. A man, wishing to ascertain the height of a tower, fixes a staff vertically in the ground at a distance of 27 ft. from the tower. Then, retiring 3 ft. farther from the tower, he sees the top of the staff in line with the top of the tower. If the observer's eye and the top of the staff are respectively 5 ft. 4 in. and 12 ft. above the ground, find the height of the tower.

14. A person due S. of a lighthouse observes that his shadow cast by the light at the top is 24 feet long. On walking 100 yards due E. he finds his shadow to be 30 feet long. Supposing him to be 6 feet high, find the height of the light from the ground.

SIMILAR POLYGONS

THEOREM 57

Similar polygons can be divided into the same number of similar triangles; and the lines joining corresponding vertices in each figure are proportional.



Let $ABCDE$, $FGHKL$ be similar polygons, the vertex A corresponding to the vertex F, B to G, and so on. Let AC, AD be joined, and also FH, FK.

It is required to prove that

(i) the ▲ ABC, FGH *are similar*; *as also the* ▲ ACD, FHK, *and the* ▲ ADE, FKL.

(ii) $$AB : FG = AC : FH = AD : FK.$$

Proof. (i) Since the polygons are similar,
the $\angle ABC$ = the $\angle FGH$, and $AB : FG = BC : GH$;

∴ the ▲ ABC, FGH are similar. *Theor.* 52.

∴ the $\angle BCA$ = the $\angle GHF$;
Also the $\angle BCD$ = the $\angle GHK$;
∴ the $\angle ACD$ = the $\angle FHK$.

Also $AC : FH = BC : GH$ (the ▲ being similar)
$= CD : HK$ (the polygons being similar).

∴ the ▲ ACD, FHK are similar. *Theor.* 52.

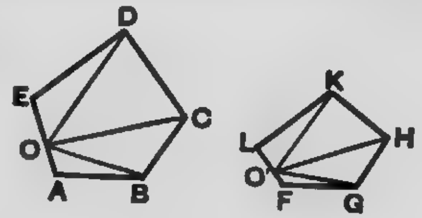
In the same way the ▲ ADE, FKL are similar.

(ii) And $AB : FG = AC : FH$, from the similar ▲ ABC, FGH;
$= AD : FK$, from the similar ▲ CAD, HFK.

Q.E.D.

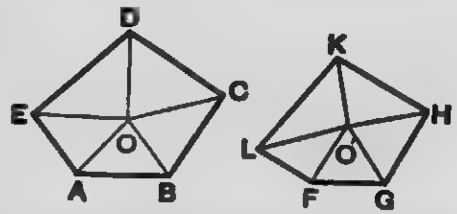
NOTE. In Theorem 57 the polygons have been divided into similar triangles by lines drawn from a pair of corresponding vertices. Other ways in which this subdivision may be made are:

(i) *By lines drawn from a pair of corresponding points on the perimeters of the figures, but not vertices.*



(ii) *By lines drawn from a pair of corresponding points within the polygons.*

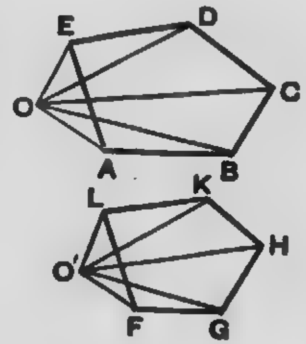
The proofs of the proposition for these cases are left as an exercise for the student.



It is well to notice also the following case in which the subdivision is made.

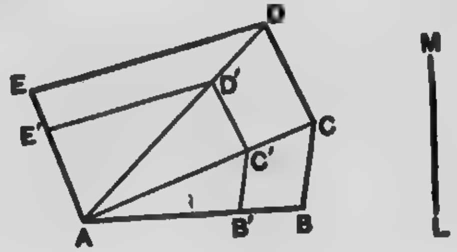
(iii) *By lines drawn from a pair of corresponding points outside the polygons.*

In this case let the student prove that the corresponding triangles are similar and note that these triangles are not parts of the polygons but that the polygon $ABCDE$ = the sum of the ▲ OAB, OBC, OCD, ODE diminished by the $\triangle$ OAE; and similarly for the polygon $FGHKL$.



Problem 37. [First Method.]

On a side of given length to draw a figure similar to a given rectilineal figure.



Let $ABCDE$ be the given figure, and LM the length of the given side; and suppose that this side is to correspond to AB.

Construction. From AB cut off AB' equal to LM.
Join AC, AD.

From B' draw $B'C'$ par[l] to BC, to cut AC at C'.
From C' draw $C'D'$ par[l] to CD, to cut AD at D'.
From D' draw $D'E'$ par[l] to DE, to cut EA at E'.
Then $AB'C'D'E'$ is the required figure.

Outline of Proof. (i) By construction the figure $AB'C'D'E'$ is equiangular to the figure $ABCDE$.

(ii) From the three pairs of similar triangles it may be shewn that

$$\frac{AB'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'E'}{DE} = \frac{E'A}{EA};$$

that is, corresponding sides of the polygons are proportional. Accordingly the figure $AB'C'D'E'$ described on a line equal to LM is similar to $ABCDE$.

Theorem 58

Any two similar rectilineal figures may be so placed that the lines joining corresponding vertices are concurrent.

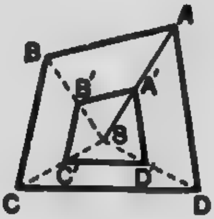


Fig. 1.

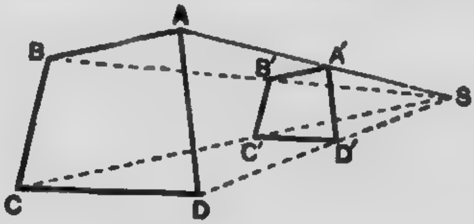


Fig. 2.

Let $ABCD$, $A'B'C'D'$ be similar figures.

Then since the $\angle B' =$ the $\angle B$, the figures can be so placed that $A'B'$, $B'C'$ are respectively par[l] to AB, BC. It follows, since the figures are equiangular to one another, that $C'D'$ is par[l] to CD, and $D'A'$ par[l] to DA.

It is required to prove that when corresponding sides of the figures are parallel, AA', BB', CC', DD' are concurrent.

Join AA'; divide it externally at S in the ratio $AB : A'B'$.

Join SB and SB'; it will be shewn that SB and SB' are in one straight line.

Proof. In the $\triangle$ $SAB, SA'B'$, since AB and $A'B'$ are par[l],

$\therefore$ the $\angle SAB =$ the $\angle SA'B'$;

and, by construction, $SA : SA' = AB : A'B'$;

$\therefore$ the $\triangle$ SAB, $SA'B'$ are similar; *Theor.* 52.

$\therefore$ the $\angle ASB =$ the $\angle A'SB'$.

Hence SB, SB' are in the same st. line;

that is, BB' passes through the fixed point S.

Similarly CC' and DD' may be shown to pass through S.

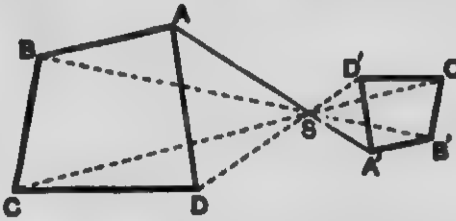
That is, AA', BB', CC', DD' are concurrent. Q.E.D.

Note. Observe that the joining lines AA', BB', CC', DD' are all divided externally at S in the ratio of any pair of corresponding sides of the given figures. S is called the centre of similarity.

NOTE. In placing the given figures so that $A'B'$, $B'C'$ are respectively parallel to AB, BC, two cases arise:

(i) $A'B'$ and AB may have *the same sense*, as in Figs. 1 and 2;

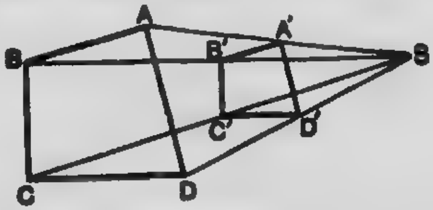
(ii) $A'B'$ and AB may have *opposite senses*, as in the Fig. below.



In the latter case it follows also that $C'D'$ is par[l] to CD, and $D'A'$ par[l] to DA, and it may be proved that AA', BB', CC', DD' are concurrent; but here S divides AA' *internally* in the ratio $AB : A'B'$.

PROBLEM 37. [Second Method.]

On a given side to draw a figure similar to a given figure.



Let $ABCD$ be the given figure, and $A'B'$ the given side; and let $A'B'$ correspond to AB.

Construction. Place $A'B'$ par[l] to AB; and join AA', BB' by lines meeting at S.

Join SC, SD.

Through B' draw $B'C'$ par[l] to BC, to meet SC at C';
through C' draw $C'D'$ par[l] to CD, to meet SD at D'.

Join $A'D'$.

Then $A'B'C'D'$ is the required figure.

The student should prove (i) that $A'B'C'D'$ is equiangular to $ABCD$, (ii) that corresponding sides of these figures are proportional. The proof is the converse of Theorem 58.

EXERCISES ON SIMILAR FIGURES

(Numerical and Graphical)

1. On a base AB, 6·5 cm. in length, draw a quadrilateral $ABCD$ from the following data:

$$\angle A = 80°, \quad \angle B = 70°, \quad AD = 4{\cdot}4 \text{ cm.}, \quad BC = 3{\cdot}2 \text{ cm.}$$

Taking any convenient point as centre of similarity, make

(i) A reduced copy of $ABCD$, such that the ratio of each side to the corresponding side of $ABCD$ is $3:4$.

(ii) An enlarged copy of $ABCD$, such that the ratio of each side to the corresponding side of $ABCD$ is $5:4$.

2. In a semi-circle drawn on a given diameter AB, inscribe a square, so that two vertices may be on the arc, and two on AB.

If $AB = 2r$, and the side of the inscribed square $= a$, shew that

$$5a^2 = 4r^2.$$

3. Draw a sector of a circle of radius 2·4″, the central angle being 60°; and inscribe a square in it.

If the radius of the sector $= r$, and the side of the square $= a$, calculate from measurements the ratio $a:r$.

4. In a sector of which the radius = 5 cm., and the central angle = 45°, inscribe a rectangle with its sides in the ratio $2:1$.

Prove that two such rectangles can be drawn, and compare by measurement their greater sides.

5. Draw a triangle ABC, making $a = 8$ cm., $b = 7$ cm., and $c = 6$ cm.

Working from the vertex A as centre of similarity, inscribe a square in the triangle, so that two of its angular points may be in the base BC, and the other two in AB, AC.

6. Draw a triangle ABC, making $a = 2{\cdot}6''$, $B = 110°$, $C = 35°$.

In the triangle ABC inscribe an equilateral triangle, having

(i) one side parallel to BC;

(ii) one side parallel to any given straight line.

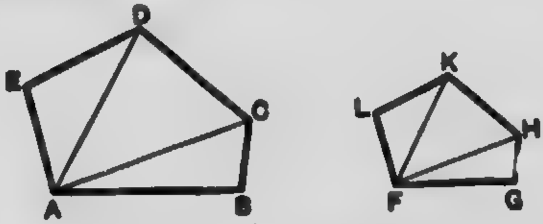
7. In a given triangle ABC inscribe a triangle similar to a given triangle DEF.

In how many ways may this be done?

8. Draw a regular hexagon $ABCDEF$ on a side of 1·2″, and in it inscribe a square having two sides parallel to AB and DE, and its vertices on the remaining sides of the hexagon.

Theorem 59. [Euclid VI. 20]

The areas of similar polygons are proportional to the squares on corresponding sides.



Let $ABCDE$, $FGHKL$ be similar polygons, and let AB FG be corresponding sides.

It is required to prove that

the polygon $ABCDE$: *the polygon* $FGHKL = AB^2 : FG^2$.

Join AC, AD, FH, FK.

Proof. Then the ▲ ABC, FGH are similar ; *Theor.* 57.
also the ▲ ACD, FHK are similar ;
and the ▲ ADE, FKL are similar.

$\therefore$ the $\triangle\, ABC$: the $\triangle\, FGH = AC^2 : FH^2$ *Theor.* 55.
$= $ the $\triangle\, ACD$: the $\triangle\, FHK$.

Similarly,
the $\triangle\, ACB$: the $\triangle\, FHK = AD^2 : FK^2$
$=$ the $\triangle\, ADE$: the $\triangle\, FKL$.

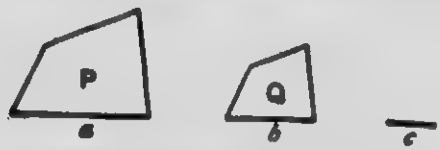
Hence $$\frac{\triangle\, ABC}{\triangle\, FGH} = \frac{\triangle\, ACD}{\triangle\, FHK} = \frac{\triangle\, ADE}{\triangle\, FKL}.$$

And in this series of equal ratios, the sum of the antecedents is to the sum of the consequents as each antecedent is to its consequent ; *Theor.* V, p. 207.

$\therefore$ the fig. $ABCDE$: the fig. $FGHKL$
$=$ the $\triangle\, ABC$: the $\triangle\, FGH$
$= AB^2 : FG^2$.

Q.E.D.

COROLLARY 1. Let a, b, c represent three lines in proportion, so that $\frac{a}{b} = \frac{b}{c}$; and consequently $b^2 = ac$.



Now suppose similar figures P and Q to be drawn on a and b as corresponding sides,

then
$$\frac{\text{Fig. } P}{\text{Fig. } Q} = \frac{a^2}{b^2} = \frac{a^2}{ac} = \frac{a}{c}.$$

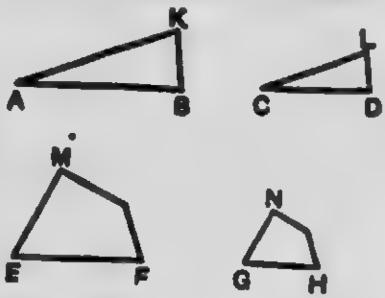
Hence if three straight lines are proportionals, and any similar figures are drawn on the first and second as corresponding sides, then

the fig. on the **first** *: the fig. on the* **second** = *the* **first** *: the* **third.**

COROLLARY 2. Let

$$AB : CD = EF : GH;$$

and let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figures MF, NH be similarly described on EF, GH.



Then since $\frac{AB}{CD} = \frac{EF}{GH}$; $\therefore \frac{AB^2}{CD^2} = \frac{EF^2}{GH^2}$.

But the fig. KAB : the fig. $LCD = AB^2 : CD^2$; *Theor.* 59.
and the fig. MF : the fig. $NH = EF^2 : GH^2$.
$\therefore$ the fig. KAB : the fig. LCD = the fig. MF : the fig. NH.

Hence if four straight lines are proportional, and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional.

EXERCISES

1. Similar figures are described on the side and diagonal of a square; prove that the ratio of their areas is 1 : 2.

2. Similar figures are described on the side and altitude of an equilateral triangle; prove that the ratio of their areas is 4 : 3.

3. The area of a regular pentagon on a side of 2·5″ is approximately 10¾ sq. in.; find the area of a similar figure on a side of 3·0″.

4. The length of a rectangular area is 10·8 metres, and the ratio of the length to the breadth is 12 : 5; find the length and breadth of a similar rectangle containing one-ninth of the area.

5. In the plan of a certain field, 1″ represents 66 yards; if the area of the plan is found to be 100 sq. in., find the area of the field in acres.

Explain why in this example the *shape* of the field is immaterial.

6. An estate is represented on a plan by a quadrilateral $ABCD$ drawn to the scale of 25″ to the mile. If $AC = 20''$, and the offsets from AC to B and D measure 24″ and 26″ respectively, find the acreage of the estate.

7. A field of 1·89 hectares is represented on a plan by a triangle whose sides measure 13 cm., 14 cm., and 15 cm. On what scale is the plan drawn?

8. A regular hexagon is drawn on a side of a cm. and a second hexagon is inscribed in it by joining the middle points of the sides in order. In like manner a third hexagon is inscribed in the second; and so on. Find the ratio of the first hexagon to the fifth.

9. Compare the area of any regular hexagon with the areas of the regular hexagons described on two unequal diagonals of the original one.

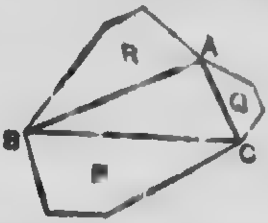
10. Compare the areas of the regular inscribed and the regular circumscribed hexagons of any circle.

11. Shew that the areas of two similar cyclic figures are proportional to the squares of the diameters of their circum-circles.

12. Two similar polygons which are equal in area are equal in all respects.

THEOREM 60. [Euclid VI. 31]

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described figures on the sides containing the right angle.



Let ABC be a right-angled triangle of which BC is the hypotenuse; and let P, Q, R be similar and similarly described figures on BC, CA, AB respectively.

It is required to prove that

the fig. R + the fig. Q = the fig. P.

Proof. Since AB and BC are corresponding sides of the similar figs. R and P,

$$\therefore \frac{\text{fig. } R}{\text{fig. } P} = \frac{AB^2}{BC^2} \quad \ldots\ldots \text{(i)} \qquad \textit{Theor. 59.}$$

In like manner,

$$\frac{\text{fig. } Q}{\text{fig. } P} = \frac{AC^2}{BC^2} \quad \ldots\ldots \text{(ii)}$$

Adding the equal ratios on each side in (i) and (ii)

$$\frac{\text{fig. } R + \text{fig. } Q}{\text{fig. } P} = \frac{AB^2 + AC^2}{BC^2}.$$

But $$AB^2 + AC^2 = BC^2; \qquad \textit{Theor. 29.}$$

$\therefore$ the fig. R + the fig. Q = the fig. P. Q.E.D.

COROLLARY. *The area of a circle drawn on the hypotenuse of a right-angled triangle as diameter is equal to the sum of the circles similarly drawn on the other sides.*

For the areas of circles are proportional to the squares on their diameters. [Page 199.]

EXERCISES

(Miscellaneous)

1. In a triangle ABC, right-angled at A, AD is drawn perpendicular to the hypotenuse. Shew that

$$\text{(i) } BA^2 = BC \cdot BD; \qquad \text{(ii) } CA^2 = CB \cdot CD.$$

Hence deduce Theorem 29, namely,

$$BC^2 = BA^2 + AC^2.$$

2. In the diagram of Theorem 60, draw AD perpendicular to BC; hence prove that, if the fig. P = the $\triangle ABC$, then

(i) the fig. Q = the $\triangle ADC$; (ii) the fig. R = the $\triangle ADB$.

3. In the diagram of Theorem 60, if $AB : AC = 8 : 5$, and if the fig. $P = 8 \cdot 9$ sq. cm., find the areas of the figs. Q and R.

4. BY and CZ are medians of the triangle ABC, and YZ is joined. Find the ratio of the triangle BGC to the triangle YGZ. [See p. 98.]

5. ABC is an isosceles triangle, the equal sides AB, AC each measuring $3 \cdot 6''$. From a point D in AB, a straight line DE is drawn cutting AC produced at E, and making the triangle ADE equal in area to the triangle ABC. If $AD = 1 \cdot 8''$, find AE.

6. AB is a diameter of a circle, and two chords AP, AQ are produced to meet the tangent at B in X and Y.

Shew that (i) the $\triangle$s APQ, AYX are similar;

(ii) the four points P, Q, Y, X are concyclic.

7. In the triangle ABC, the angle A is externally bisected by a line which meets the base produced at D and the circum-circle at E; shew that

$$AB \cdot AC = AE \cdot AD.$$

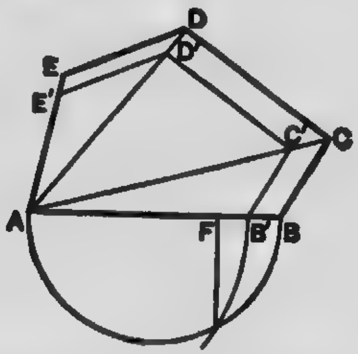
8. Draw an isosceles triangle equal in area to a triangle ABC, and having its vertical angle equal to the angle A.

9. On a given base draw an isosceles triangle equal in area to a given triangle ABC.

10. Any regular polygon inscribed in a circle is the geometric mean between the inscribed and circumscribed regular polygons of half the number of sides.

PROBLEM 38

To draw a figure similar to a given rectilineal figure, and equal to a given fraction of it in area.



Let $ABCDE$ be the given figure, to which a similar figure is to be drawn, having its area a given fraction (say *three-fourths*) of that of the fig. $ABCDE$.

Construction. Make AF *three-fourths* of AB. *Prob.* 7.

From AB cut off AB' the mean proportional between AF and AB. *Prob.* 39. *Note.*

On AB' draw the fig. $AB'C'D'E'$ similar to the fig. $ABCDE$. *Prob.* 40.

Then the fig. $AB'C'D'E' = \frac{3}{4}$ of the fig. $ABCDE$.

Proof. By construction, $AB'^2 = AF \cdot AB$.

Now the figs. $ABCDE$, $AB'C'D'E'$ are similar, and AB, AB' are corresponding sides;

$$\therefore \frac{\text{fig. } AB'C'D'E'}{\text{fig. } ABCDE} = \frac{AB'^2}{AB^2} \qquad \textit{Theor. } 59.$$

$$= \frac{AF \cdot AB}{AB^2}$$

$$= \frac{AF}{AB} = \frac{3}{4}.$$

EXERCISES

1. Divide a triangle ABC into two parts of equal area by a line XY drawn parallel to the base BC and cutting the other sides at X and Y.

Find (i) by calculation, (ii) by measurement, the ratio $AX : AB$.

2. Divide a triangle ABC into three parts of equal area by lines PQ, XY drawn parallel to the base BC. If P and X lie in AB, prove that

$$\frac{AP}{1} = \frac{AX}{\sqrt{2}} = \frac{AB}{\sqrt{3}}.$$

Hence shew how a triangle may be divided into n equal parts by lines drawn parallel to one side.

3. Draw a rectangle of length 8 cm., and breadth 5 cm. Then draw a similar rectangle of one-third the area.

Measure its length to the nearest millimetre, and verify your result by calculation.

4. Draw a quadrilateral $ABCD$ from the following data:

the $\angle A = 90°$; $AB = BC = 8$ cm.; $AD = DC = 6$ cm.

Draw a similar quadrilateral to contain an area of 36 sq. cm., and find to the nearest millimetre the length of the side corresponding to AB.

5. Divide a circle of radius 3″ into three equal parts by means of two concentric circles.

6. *Draw a rectilineal figure equal in area to a given figure E, and similar to a given figure S.* [Euclid VI. 25.]

[First replace the given figures E and S by equivalent squares (see Problems 19 and 33). Let the sides of these squares be a and b respectively, and let s be one of the sides of S.

Find p, a fourth proportional to b, a, s, so that $b : a = s : p$.

On p draw a figure P similar to the figure S, so that p and s are corresponding sides. Then P is the figure required;

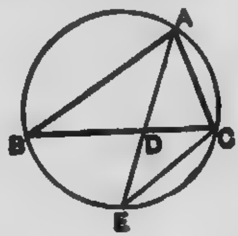
for $$\frac{P}{S} = \frac{p^2}{s^2} = \frac{a^2}{b^2} = \frac{E}{S}.$$

$\therefore$ the fig. P = the fig. E.]

MISCELLANEOUS THEOREMS

*Theorem 61

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.



Let ABC be a triangle, having the $\angle\ BAC$ bisected by AD.

It is required to prove that

the rect. AB, AC = *the rect.* BD, DC + *the sq. on* AD.

Suppose a circle circumscribed about the $\triangle\ ABC$; and let AD be produced to meet the $\bigcirc^{ce}$ at E.

Join EC.

Proof. Then in the $\triangle s$ BAD, EAC,

because the $\angle\ BAD$ = the $\angle\ EAC$,

and the $\angle\ ABD$ = the $\angle\ AEC$ in the same segment;

$\therefore$ the remaining $\angle\ BDA$ = the remaining $\angle\ ECA$;

that is, the $\triangle s$ BAD, EAC are equiangular to one another;

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}. \qquad \textit{Theor. 50.}$$

Hence $\quad AB\cdot AC = AE\cdot AD = (AD + DE)\,AD$

$\qquad\qquad = AD^2 + AD\cdot DE.$

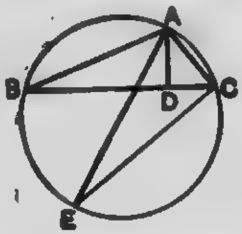
But $\quad AD\cdot DE = BD\cdot DC$; *Theor.* 56.

$\therefore$ the rect. AB, AC = the rect. BD, DC + the sq. on AD.

Q.E.D.

* Theorem 62

If from the vertical angle of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.



In the $\triangle ABC$, let AD be the perp. from A to the base BC; and let AE be a diameter of the circum-circle.

It is required to prove that

the rect. AB, AC = *the rect.* AE, AD.

Join EC.

Proof. Then in the $\triangle$s BAD, EAC,
the rt. angle BDA = the rt. angle ECA, in the semi-circle ECA,

and the $\angle ABD$ = the $\angle AEC$, in the same segment.

$\therefore$ the remaining $\angle BAD$ = the remaining $\angle EAC$;
that is, the $\triangle$s BAD, EAC are equiangular to one another.

$\therefore AB : AE = AD : AC$; *Theor.* 50.

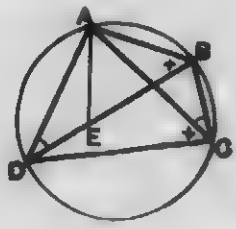
Hence the rect. AB, AC = the rect. AE, AD. Q.E.D.

Note. Let a, b, c denote the sides of the $\triangle ABC$, R its circum-radius, and p the perp. AD.

Then since
$$AE \cdot AD = AB \cdot AC$$
$$2R \cdot p = cb.$$
$$\therefore R = \frac{bc}{2p}$$
$$= \frac{abc}{2ap} = \frac{abc}{4\Delta}.$$

THEOREM 63. [Ptolemy's Theorem]

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let $ABCD$ be a quadrilateral inscribed in a circle and let AC, BD be its diagonals.

It is required to prove that

the rect. AC, BD = *the rect.* AB, CD + *the rect.* BC, DA.

Make the $\angle DAE$ equal to the $\angle BAC$;

to each add the $\angle EAC$,

then the $\angle DAC$ = the $\angle EAB$.

Proof. Since the $\angle EAB$ = the $\angle DAC$,

and the $\angle ABE$ = the $\angle ACD$ in the same segment;

$\therefore$ the $\triangle EAB, DAC$ are equiangular to one another;

$\therefore BA : CA = BE : CD$; *Theor.* 50.

hence $AB \cdot CD = AC \cdot BE$. . . . . (i)

Again in the $\triangle DAE, CAB$,

the $\angle DAE$ = the $\angle CAB$,

and the $\angle ADE$ = the $\angle ACB$, in the same segment;

$\therefore$ the $\triangle DAE, CAB$ are equiangular to one another.

$\therefore DA : CA = DE : CB$;

hence $BC \cdot DA = AC \cdot DE$. . . . . (ii)

Adding the equal rectangles on each side in (i) and (ii)

$$AB \cdot CD + BC \cdot DA = AC \cdot BE + AC \cdot DE$$
$$= AC(BE + DE)$$
$$= AC \cdot BD.$$

Q.E.D.

EXERCISES

1. ABC is an isosceles triangle, and on the base, or base produced, any point X is taken; shew that the circumscribed circles of the triangles ABX, ACX are equal.

2. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn perpendicular to AB, AC respectively, and intersecting at D; shew that

$$BC \cdot AD = 2AB \cdot DB.$$

3. If the diagonals of a quadrilateral inscribed in a circle are at right angles, the sum of the rectangles contained by the opposite sides is double the area of the figure.

4. $ABCD$ is a quadrilateral inscribed in a circle, and the diagonal BD bisects AC; shew that

$$AD \cdot AB = DC \cdot CB.$$

5. If the vertex A of a triangle ABC is joined to any point in the base, it will divide the triangle into two triangles such that their circumscribed circles have radii in the ratio of AB to AC.

6. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

7. Two triangles of equal area are inscribed in the same circle; shew that the rectangle contained by any two sides of the one is to the rectangle contained by any two sides of the other as the base of the second is to the base of the first.

8. P is a point on the arc BC of the circum-circle of an equilateral triangle ABC. If P is joined to A, B, and C, shew that

$$PB + PC = PA.$$

9. $ABCD$ is a quadrilateral inscribed in a circle, and BD bisects the angle ABC; if the points A and C are fixed on the circumference of the circle, and B is variable in position, shew that

$AB + BC : BD$ is a constant ratio.

10. From the formula $R = \frac{abc}{4\Delta}$ (see NOTE, p. 254) find the value of R when the sides of the triangle are as follows:

(i) 21″, 20″, 13″; (ii) 30 ft., 25 ft., 11 ft.

Draw to a convenient scale and check your work by measurement.

MISCELLANEOUS EXAMPLES

PARTS I–IV

1. The bisector of the angle P of the triangle PQR meets QR at S and QR is produced to T. Prove the sum of the angles PQR and PRT equals twice the angle PSR.

2. L and M are the middle points of the sides PQ, PR of the $\triangle PQR$. RL and QM are produced to T and S so that $RL = LT$ and $QM = MS$. Prove that T, P, S are collinear and that $PT = PS$.

3. In the isosceles $\triangle PQR$, $PQ = PR$. PS and PT are equal parts cut off from PQ, PR respectively. QT, RS intersect at O. Prove ▲ TOS, QOR isosceles.

4. A st. line PR is bisected at Q. From P and R PT, RS are drawn perpendicular to any other st. line and QS, QT joined; prove $\triangle QTS$ isosceles.

5. PQR is a $\triangle$. PS is $\perp$ QR and PT bisects angle QPR. Prove angle SPT = half the difference of the angles Q and R.

6. Find a point such that its distances from two given intersecting straight lines shall be equal to two given lengths.

7. G is any point in the base EF of the isosceles $\triangle DEF$. DG is joined and bisected at H. Prove $HF > HG$.

8. The vertical $\angle A$ of the $\triangle ABC$ is bisected by AD which meets the base BC at D. DM, DN drawn $\parallel$ to AB, AC resp. meet AC in M and AB in N. Prove the four sides of figure $ANDM$ equal.

9. The base BC of the $\triangle ABC$ is produced to D. BO bisecting $\angle ABC$ and CO bisecting $\angle ACD$ meet at O. Prove $\angle BOC = \frac{1}{2} \angle A$.

10. AD joins the vertex A of the triangle ABC to the middle point D of BC. Shew that $AD >, = >$ or $< BD$ according as $\angle BAC$ is acute, right, or obtuse.

11. BC is the base of an isosceles triangle ABC. A circle with centre C and radius CB cuts AB, AC in D and E resp. Shew that DE is parallel to the bisector of $\angle B$.

12. The quadrilateral formed by the bisectors of the angles of any quadrilateral is cyclic.

13. PQ and RS are two equal straight lines not in the same straight line. Find a point T so that the $\triangle TPQ = \triangle TRS$.

14. $PQRS$ is a parallelogram. DE drawn $\parallel$ PR meets SP, SR produced if necessary at D and E. Prove $\triangle QDP = \triangle QER$.

15. Trisect a parallelogram by st. lines through a vertex.

16. P and Q are two fixed points. Find a point O such that $OP^2 + OQ^2$ may be a minimum.

17. $PQRS$ is a parallelogram. PT is drawn to any point T in QR and O is any point in PT. Prove $\triangle QOR = \triangle TOS$.

18. If two chords of a circle intersect at right angles, the sum of the squares on their segments equals the square on a diameter.

19. Find a point within a given triangle at which the three sides subtend equal angles. When is the solution possible?

20. Through an intersection of two given circles draw the greatest possible st. line terminated by the two circumferences.

21. Describe a circle of given radius to touch two given circles.

22. Describe a circle of given radius to touch two given intersecting st. lines.

23. From a given point P without a given circle draw a secant PQR such that $PQ = QR$.

24. From the extremities of the diameter of a circle perpendiculars are drawn to any chord. Shew that the centre is equally distant from the feet of the perpendiculars.

25. Draw a tangent to a circle which shall bisect a given parallelogram which is outside the circle.

26. Describe a circle with given radius to touch a given st. line and have its centre in another given st. line.

27. Describe a circle with given radius to pass through a given point and touch a given st. line.

28. Describe a circle with given radius to touch a given circle and a given st. line.

29. AD and AE bisect the interior and exterior angles at A of $\triangle ABC$, and meet BC at D and E; and O is the middle point of BC. Prove $OC^2 = OD \cdot OE$.

30. In a given circle inscribe a triangle whose sides are parallel to three given st. lines.

31. Two circles whose centres are A and B touch externally at P, and CPD is drawn meeting the circles in C and D. Shew that the triangles APD, CPB are equal in area.

32. Construct a triangle equiangular to a given triangle and having a given circle for one of its escribed circles.

33. Construct a triangle, given the base, the vertical angle, and the radius of the inscribed circle.

34. If two circles intersect and through a point on their common chord produced two secants are drawn, one to each circle, the four points of section of the secants with the circles are concyclic.

35. If ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC, shew that

(i) $BC^2 : BA^2 = BC : BD$;
(ii) $BC^2 : CA^2 = BC : CD$.

Hence deduce $BC^2 = BA^2 + AC^2$.

36. A triangle ABC is bisected by a straight line XY drawn parallel to the base BC. Determine the ratio $AX : AB$.

Hence bisect a triangle by a line drawn parallel to the base.

37. If two circles have external contact at A, and a common tangent, touching them at B and C, meets the line of centres at S,

$$\triangle SBA : \triangle SAC = SB : SC.$$

38. Two circles intersect at A and B, and at A tangents are drawn, one to each circle, meeting the circumferences at C and D. If AB, CB, and BD are joined, shew that

$$\triangle CBA : \triangle ABD = CB : BD.$$

39. DEF is the pedal triangle of the triangle ABC; prove that

$$\triangle ABC : \triangle DBF = AB^2 : DB^2;$$

$$\text{fig. } AFDC : \triangle DBF = AD^2 : BD^2.$$

40. In a given triangle ABC a second triangle is inscribed by joining the middle points of the sides. In this inscribed triangle a third is inscribed in like manner, and so on. What fraction is the fourth triangle of the triangle ABC?

41. A semi-circle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P. Shew that

$$AB^2 = AC \cdot AP + BD \cdot BP.$$

42. Two circles intersect at B and C, and the two direct common tangents AE and DF aré drawn; if the common chord is produced to meet the tangents at G and H, shew that

$$GH^2 = AE^2 + BC^2.$$

43. If from an external point P, a secant PCD is drawn to a circle and PM is perpendicular to a diameter AB, shew that

$$PM^2 = PC \cdot PD + AM \cdot MB.$$

44. Two circles whose centres are C and D intersect at A and B; and a straight line PAQ is drawn through A and terminated by the circumferences: prove that

(i) the $\angle\, PBQ$ = the $\angle\, CAD$;
(ii) the $\angle\, BPC$ = the $\angle\, BQD$.

45. AB is a given diameter of a circle, and CD is any chord parallel to AB; if X is any point in AB,

$$XC^2 + XD^2 = XA^2 + XB^2.$$

46. If the opposite sides of a cyclic quadrilateral are produced to meet, the bisectors of the angles so formed are perpendicular.

47. Given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex on the base: construct the triangle.

48. A, B, C are three points in order in a straight line: find a point P in the straight line such that $PA : PB = PB : PC$.

49. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

50. Given the base, and the position of the bisector of the vertical angle: construct the triangle.

ANSWERS TO NUMERICAL EXERCISES

Since the utmost care cannot ensure absolute accuracy in graphical work, results so obtained are likely to be only approximate. The answers here given are those found by calculation, and being true so far as they go, furnish a standard by which the student may test the correctness of his drawing and measurement. Results within one per cent of those given in the Answers may usually be considered satisfactory.

Exercises. Page 15

1. 30°; 126°; 261°; 85°. 11 min.; 37 min.
2. 112½°; 155°; 5 hrs. 45 min. *3.* 50°; 8 hrs. 40 min.
4. (i) 145°, 35°, 145°. (ii) 55°, 55°. 86°, 94°.

Exercises. Page 27

1. 68°, 37°, 75° v. nearly. *2.* 6·0 cm. *4.* 2·2″, 50°, 73° nearly.
5. 37 ft. *6.* 101 metres. *7.* 27 ft. *8.* 424 yds., nearly; N. W.
9. 281 yds., 155 yds., 153 yds. *10.* 214 yds.

Exercises. Page 41

1. 125°, 55°, 125°. *12.* 15 secs., 30 secs.

Exercises. Page 43

3. 21°. *4.* 27°. *5.* 92°, 46°. *6.* 67°, 62°.

Exercises. Page 45

1. 30°, 60°, 90°. *2.* (i) 36°, 72°, 72°; (ii) 20°, 80°, 80°.
3. 40°. *4.* 51°, 111°, 18°. *5.* (i) 34°; (ii) 107°.
6. 68°. *7.* 120°. *8.* 36°, 72°, 108°, 144°.
9. 165°. *11.* 5, 15.

Exercises. Page 47

2. (i) 45°; (ii) 36°. *3.* (i) 12; (ii) 15.

MICROCOPY RESOLUTION TEST CHART

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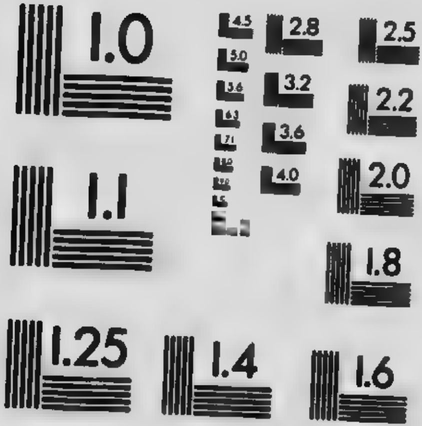

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Exercises. Page 54

4. (i) 81°; *c.* (ii) 55°.

10.

Degrees	15°	30°	45°	60°	75°
Cm.	41	4·6	5·7	8·0	15·6

11.

Degrees	0°	30°	60°	90	120°	150°	180°
Cm.	1·0	2·0	3·6	5·0	6·1	6·8	7·0

12. 37 ft. *13.* 112 ft. *14.* 346 yds. 693 yds.

Exercises. Page 61

14. 54°, 72°, 54°. *15.* 36°. *16.* 4.
18. (i) 16; (ii) 45°; (iii) $11\frac{1}{4}$° per sec.

Exercises. Page 68

2. 6·80 cm. *3.* 2·24″. *4.* 0·39. *5.* 2·54. *8.* 10·6 cm.
9. 3·35″. *10.* 20 miles; 12·6 km.
11. 147 miles; 235 km. 1 cm. represents 22 km.
12. 1″ represents 15 mi.; 1″ represents 20 mi.

Exercises. Page 79

3. 0·43 in. *4.* 1·3 cm. *5.* 2·4″.

Exercises. Page 84

1. 4·3 cm., 5·2 cm., 6·1 cm. *2.* 1·10. *3.* 200 yards.
4. 65°, 77 m., 61 m., 56 m. *5.* 6·04 knots. S, 15° E, nearly.
6. Results equal. 9 cm. *7.* 4·3 cm.; 9·8 cm., 60°; 120°.
8. (i) One solution; (ii) two; (iii) one, right-angled; (iv) impossible.
9. 380 yds. *10.* 6·5 cm. *11.* 6·9 cm.
12. Two solutions; 10·4 cm. or 4·5 cm. *16.* 2·8 cm., 4·5 cm., 5·3 cm.
18. 5·8 cm., 4·2 cm. *19.* 7 cm., 8 cm.

Exercises. Page 89

1. 60°, 120°. *2.* 3·54″. *3.* 2·12″. *4.* 4·4 cm.
5. 16·4 cm., 3·4%. *6.* 90°. *7.* (i) 4·25″; (ii) $B = D = 90°$.

Exercises. Page 104

1. 6 sq. in. *2.* 6 sq. in. *3.* 2·80 sq. in. *4.* 3·50 sq. in.
5. 3·30 sq. in. *6.* 3·36 sq. in. *7.* 198 sq. m. *8.* 42 sq. ft.
9. 10,000 sq. m. *10.* 110 sq. ft. *11.* 5 cm. *12.* 2·6 in.
14. 900 sq. yds.; 48 yds.; 4·8″. *15.* 11,700 sq. m.
16. 1 cm. = 10 yds. *17.* 1·6″. *18.* 600 sq. ft. *19.* 1154 sq. ft.
20. 100 sq. ft. *21.* 156 sq. ft. *22.* 110 sq. ft.
23. 288 sq. ft. *24.* 72 sq. ft. *25.* 75 sq. ft.

Exercises. Page 107

1. (i) 22 cm.; (ii) 3·6″. *2.* 3·4 sq. in. *3.* 574·5 sq. in.
4. 1·5″. *5.* 1·93″, 75°.

Exercises. Page 109

1. (i) 180 sq. ft.; (ii) 8·4 sq. in.; 1 hectare.
2. (i) 13·44 sq. cm.; (ii) 15·40 sq. cm.; (iii) 20·50 sq. cm.
3. 15 sq. cm. *4.* 6·3 sq. in.
5. (i) 8″; (ii) 13 cm. *6.* 3·36 sq. in.

Exercises. Page 112

1. 11,400 sq. yds. *2.* 6312 sq. m.
3. 2.4 cm.; 5.1 cm. *4.* 2.04″; 2.20″.

5.

Angle	0°	30°	60°	90°	120°	150°	180°
Area in sq. cm.	0	7·5	13·0	15·0	13·0	7·5	0

Exercises. Page 113

1. 66 sq. ft. *2.* 84 sq. yds. *3.* 126 sq. m.
4. 132 sq. cm. *5.* 180 sq. ft. *6.* 306 sq. m.

Exercises. Page 116

1. 6 sq. in. *2.* 170 sq. ft. *3.* 615 sq. m. *4.* 8·4 sq. in.
5. 31·2 sq. cm. *6.* 5·20 sq. in. *7.* 24 sq. cm.

Exercises. Page 117

1. (i) 25·5 sq. cm.; (ii) 15·6 sq. cm.
2. (i) 8·95 sq. in.; (ii) 9·5 sq. in. *3.* 12,500 sq. m.

Exercises. Page 118

4. 3·3 sq. in. *5.* 7·5 cm. *6.* 3·6 sq. in.

Exercises. Page 123

1. (i) 5 cm.; (ii) 6·5 cm.; (iii) 3·7″. *2.* (i) 1·6″; (ii) 2·8 cm.
3. 41 ft. *4.* 65 miles *5.* 6·1 km. *6.* 16 ft.
7. 48 m. *8.* 25 miles. *9.* 73 m. *10.* 62 ft.

Exercises. Page 125

10. (i) and (iii). *11.* 2·83″. *12.* 4·24 cm.; 18 sq. cm.
13. 70·71 sq. m. *14.* $p = 6{\cdot}93$ cm.
16. (i) 20 cm.; 15 cm.; (ii) 40 cm.; 39 cm.
17. 35 cm.; 12 cm.; 306 sq. cm.
18. (i) 36 sq. in.; (ii) 90 sq. ft.; (iii) 126 sq. cm.; (iv) 240 sq. yds.
19. 5·1 cm. nearly.

Exercises. Page 132

1. 630 sq. cm.; 15 cm.

Exercises. Page 134

2. 8·5 cm.; 90°. *3.* A circle of radius 6 cm.
4. 5·20″. *6.* 0·25″.

Exercises. Page 136

1. 7·1 cm. *4.* 4·0 cm. *5.* 1·6″. *6.* 3·1 cm.; 15·6 sq. cm.

Exercises. Page 140

1. 23·90 sq. cm. *2.* 8·40 sq. in.
3. 27·52 sq. cm. *4.* 129,800 sq. m.

Exercises. Page 149

1. 5 cm. *2.* 24″. *3.* 0·6″, 0·8″. *4.* $\sqrt{7} = 2{\cdot}6$ cm.
5. 1 ft. *6.* 0·6 sq. in. *7.* 0·8″.

Exercises. Page 153

1. 1·7″. *2.* $3\sqrt{2} = 4{\cdot}2$ cm. *3.* $2\sqrt{3} = 3{\cdot}5$ cm.
4. 17″. *6.* 5 cm.

Exercises. Page 155

6. 4 cm. *7.* 1·3″.

Exercises. Page 157

2. 1·85″. *3.* 1·62″.

Exercises. Page 160

5. 51″. *6.* 1.6″; 1.5″; 0.6″.

Exercises. Page 163

1. 74°, 148°, 16°. *2.* 115°, 230°. *3.* 55°, 8°, 47°.

Exercises. Page 172

1. 8·0 cm. *2.* 0·6″. *3.* 8·7 cm. *4.* 12″, 67°. *5.* 2·5″.

Exercises. Page 174

3. 3 cm. and 17 cm.

Exercises. Page 176

1. 72°, 108°, 108°.

Exercises. Page 180

2. 1·6″. *3.* 1·7″. *4.* 1·98″, 1·6″.

Exercises. Page 193

[illegible] [illegible] cm., 4·6 cm., 6·9 cm. *3.* 1·39″.
4. [illegible]·9 cm.; 20·78 sq. cm. *7.* 3·2 cm.

Exercises. Page 194

1. 2·12″; 4·50 sq. in. *4.* 8·5 cm. *5.* 2·0″.

Exercises. Page 195

4. 128$\frac{4}{7}$°; 1·73″.

Exercises. Page 196

1. 3·46″; 4·00″. *2.* 259·8 sq. cm.
4. (i) 41·57 sq. cm.; (ii) 77·25 sq. cm.

Exercises. Page 200

1. (i) 28·3 cm.; (ii) 628·3 cm. *2.* (i) 16·62 sq. in.; (ii) 352·99 sq. in.
3. 11·31 cm.; 10·18 sq. cm. *4.* 56 sq. cm. *5.* 43·98 sq. in.
7. 30·5 sq. cm. *8.* 8·9″. *9.* 4″; 3″. *10.* 12·57 sq. in.

Exercises. Page 209

1. (i) 35; (ii) 8; (iii) *a*.
3. 4·0″, 5·6″. *4.* 16·5 cm., 12·0 cm.
5. 4·0 cm., 2·4 cm.; 16·0 cm., 9·6 cm.

Exercises. Page 214

1. (i) each = 3 : 2; (ii) each = 5 : 3; (iii) each = 5 : 2.
2. (i) 1·4″; (ii) 0·8″; (iii) 6·4 cm., 2·4 cm.
3. (i) 5·6 cm.; (ii) 7·7 cm., 2·8 cm.

Exercises. Page 215

1. 0·9″, 0·6″; 4·5″, 3·0″; 3 : 2.
2. 2·0 cm., 1·5 cm.; 14·0 cm., 10·5 cm.

Exercises. Page 217

1. 10·5 sq. in. *2.* 3·0 cm. *3.* 64 sq. cm.
4. 11·0″. *5.* 33·9 acres.

Exercises. Page 222

1. (i) 1·2″; (ii) 2·0″; (iii) 7·7 cm. *2.* (i) 2·1″; (ii) 6·3 cm.
3. $QB = 3{\cdot}5''$, $BR = 2{\cdot}5''$. *4.* 3·2 cm., 4·2 cm.
5. 2·1″, 1·8″. *6.* 5 ft., 12$\frac{3}{4}$ ft., 9$\frac{1}{4}$ ft.
7. 1·2″, 1·3″, 1·95.″ *8.* 5$\frac{3}{5}$ cm.
9. 0·8 cm., 1·4 cm., 2·1 cm.

Exercises. Page 230

1. $\frac{1}{9}$. *2.* 20 sq. ft. *3.* 10 sq. cm. *4.* 7 : 5. *5.* 5·6″.

Exercises. Page 234

1. 26″. *2.* 48 ft.; 8 ft. *3.* 2 cm.; 32 cm.
4. 3·6″. *5.* 8100 miles; 10 miles.

Exercises. Page 238

1. (i) 1·0″; (ii) 0·9″; (iii) 6·0 cm.
2. 1·4″, 0·6″; 3·5″, 1·5″. *3.* (i) 2·0; (ii) 2·8; (iii) 20.
4. 1·6 cm., 2·4 cm., 3·2 cm. *5.* 1·8″, 1·2″, 0′9″. *6.* 2·7″.
7. (i) 1·73; (ii) 3·16; (iii) 1·67. *8.* (i) 3; (ii) 3·21; (iii) 2·26.
9. (i) 1·2″, 1·6″, 2·0″; (ii) 3·0 cm., 3·6 cm., 4·5 cm.; (iii) 2·5 cm., 4·3 cm., 5·0 cm.; (iv) b = 3·4″, c = 2·1″, nearly.
10. 140 m., 160 m.; 125 m. *11.* 24 ft., 2 ft. 4 in.
12. 60 ft. *13.* 72 ft. *14.* 106 ft.

Exercises. Page 245

3. 0·52. *5.* 31 : 28, nearly.

Exercises. Page 248

3. 15·48 sq. in. *4.* 3·6 m., 1·5 m.
5. 90 acres. *6.* 512 acres.
7. 1 cm. represents 15 metres.

Exercises. Page 250

3. 2·5 sq. cm., 6·4 sq. cm. *4.* 4 : 1.
5. 7·2″. *8.* 6·2 cm., 3·8 cm.

Exercises. Page 252

1. 1 : $\sqrt{2}$. *3.* 4·6 cm. *4.* 6·9 cm.

Exercises. Page 256

10. (i) $10\frac{3}{8}''$; (ii) $15\frac{1}{4}$ ft.